THE COMPACT OPEN TOPOLOGY FOR A SPACE OF RELATIONS AND CERTAIN MONOTONE RELATIONS WHICH PRESERVE ARCS, PSEUDOCIRCLES AND TREES

By

JANE MAXWELL DAY

A DISSERTATION PRESENTED TO THE GRADUATE COUNCIL OF
THE UNIVERSITY OF FLORIDA
IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE
DEGREE OF DOCTOR OF PHILOSOPHY

UNIVERSITY OF FLORIDA

April, 1964

ACKNOWLEDGMENTS

The author is especially indebted to her chairman, Dr. W. L. Strother, for his instruction, constant encouragement and guidance throughout her doctoral study, as well as during the writing of this paper; she also wishes to thank Dr. J. E. Maxfield for his patience and encouragement during her doctoral work, and Dr. A. D. Wallace for his time and suggestions given during the preparation of this paper.

She is deeply grateful to her husband, Walter, for his understanding, encouragement, generosity and the extra measure of patience he continually offered. Without his support, her graduate work could not have been done.

TABLE OF CONTENTS

INTRODUCTION

The work in Part I is a study of relation spaces with the compact open topology. One result is: when X is Hausdorff and $\underline{R}$ is the real numbers, a certain space of relations, each in X x $\underline{R}$, forms a semialgebra.

In Part II are theorems which generalize to monotone relations various known theorems about monotone single-valued functions. For example, if X is Hausdorff and I is an arc, sufficient conditions are given on a monotone relation in I x X to imply that X is also an arc. This theorem has as a special case the well-known result that a continuous, monotone image of an arc in a Hausdorff space is again an arc.

PART I

SECTION 1. Background and definitions.

We will say $f: X \to Y$ is a *function* from X to Y iff for each x in X, f(x) is a single point of Y. If X is a set and $\mathcal{T}$ is a collection of subsets of X, $\mathcal{T}$ will be called a *topology* for X iff the empty set and X are in $\mathcal{T}$ and any union or finite intersection of elements of $\mathcal{T}$ is in $\mathcal{T}$. A *topological space*, or just *space*, is a pair $(X, \mathcal{T})$, where X is a nonnull set X and $\mathcal{T}$ is a topology for X. We will often say X is a space without specifying a topology for X, if no confusion can arise. If X is a space and $\mathcal{S}$ is any collection of subsets of X, it is simply a logical exercise to prove that the collection $\mathcal{T}$ of all unions and finite intersections of elements of $\mathcal{S}$ is a topology for X. We will say $\mathcal{S}$ *generates* $\mathcal{T}$, or $\mathcal{S}$ is a *subbasis* for $\mathcal{T}$.

Let *Y^X* be the set of all functions from X to Y, and let $B(K,W) = \{f \in Y^X \mid f(K) \subset W\}$. Let X and Y be spaces and let $\mathcal{C}$ be the topology for Y^X generated by $\{B(K,W) \mid K$ is compact in X and W is open in Y$\}$. $\mathcal{C}$ is known as the *compact open topology* for Y^X. Let $F \subset Y^X$, let F have topology $\mathcal{T}$ and let $K \subset X$; define $p: F \times K \to Y$ by $p(f,x) = f(x)$. p is known as the *evaluation function*, and $\mathcal{T}$ is said to be *jointly continuous on K* iff p is continuous.

The compact open topology for Y^X, $\mathcal{C}$, is characterized by these facts: each topology for Y^X which is jointly continuous on compacta

is larger than $\mathcal{C}$; and when X is Hausdorff or regular and F is a subset of Y^X containing only functions continuous on the compacta of X, then $\mathcal{C} | F$ is jointly continuous on compacta. These theorems can be found in [8]. $\mathcal{C} | F$ denotes $\{C \cap F \mid C \in \mathcal{C}\}$, which is a topology for F.

Before we can define the compact open topology for a space of relations, we need several more definitions and a lemma, 1.1.

A relation from X to Y is defined to be a set $R \subset X \times Y$. For each x in X, xR is defined to be $\pi_2(R \cap (\{x\} \times Y))$, and for each y in Y, Ry is defined to be the set $\pi_1(R \cap (X \times \{y\}))$. For subsets A of X and B of Y, we define AR to be $\bigcup_{a \in A} aR$ and RB to be $\bigcup_{b \in B} Rb$.

Notice that $\pi_1(R) = RY$ and $\pi_2(R) = XR$, and, if $\square$ represents the null set, $\square R$ and $R\square$ are $\square$.

All authors of theorems about multi-valued functions seem to agree to define $F{:}X \longrightarrow Y$ to be a multi-valued function from X to Y iff F(x) is a nonnull subset of Y for each x in X. Using this definition, a relation $R \subset X \times Y$ such that $X = RY$ is the graph of a multi-valued function $F{:}X \longrightarrow Y$ defined by $F(x) = xR$, and conversely if $F{:}X \longrightarrow Y$ is a multi-valued function, the graph of F is a relation $R \subset X \times Y$ such that $X = RY$.

We will not write in multi-function terminology, but we will restrict our attention to relations "defined on all of X," that is, relations $R \subset X \times Y$ such that $X = RY$. Such a relation might be called a functional relation since there is a unique multi-function associated with it.

Since many readers may be familiar with multi-function rather than relation theory terminology, we will attempt, when defining a property for relations, to mention by what other names it is known in the literature.

If X and Y are spaces, we will let S(X,Y) = $\{R \subset X \times Y \mid X = RY\}$.

The set of all nonnull subsets of a set Z will be denoted by 2^Z. If Y is a set and $U \subset Y$, let $2_U = \{A \in 2^Y \mid A \cap U \neq \square\}$.

When Y is a space, $\{2^U$ and $2_U \mid U$ open in $Y\}$ generates a topology for 2^Y, and throughout this paper when we speak of the space 2^Y, we mean 2^Y with this topology. It appears to have originated with Vietoris in [17], who named it the finite topology for 2^Y. Michael uses this name in [11]. Choquet defined it essentially the same way as Vietoris in [3]. Their definitions are easily shown to be equivalent to Frink's neighborhood topology, defined in [5]. This is the name Strother uses for it, in [15] and other papers.

If $f \in (2^Y)^X$, let us call $\cup \{x \times f(x) \mid x \in X\}$, where f(x) is considered as a subset of Y, the graph of f in X x Y.

SECTION 2. The compact open topology and joint continuity for a relation space.

The following lemma is due to many authors.

(1.1) <u>Lemma</u>. If $R \varepsilon S(X,Y)$, there is a unique $f \varepsilon (2^Y)^X$ such that R is the graph of f in X x Y. Conversely, if $f \varepsilon (2^Y)^X$, the graph of f in X x Y is unique and is an element of $S(X,Y)$.

Proof: Let $R \varepsilon S(X,Y)$. By definition then, $X = RY$: i.e., for each $x \varepsilon X$, there is some $y \varepsilon Y$ such that $x \varepsilon Ry$. Hence $xR \neq \square$ for any $x \varepsilon X$. Define $f\colon X \longrightarrow 2^Y$ by $f(x) = xR$ for each $x \varepsilon X$. $xR \neq \square$ so $xR \varepsilon 2^Y$, hence $f \varepsilon (2^Y)^X$. The graph in X x Y of f is $\cup \{\{x\} \times f(x) \mid x \varepsilon X\}$ which is by definition, $\cup \{\{x\} \times xR \mid x \varepsilon X\}$, which is just R. Clearly, different functions have different graphs, so f is unique.

Conversely, let $f \varepsilon (2^Y)^X$, and define $R \subset X \times Y$ thus: $R = \cup \{\{x\} \times f(x) \mid x \varepsilon X\}$, where $f(x)$ is considered as a subset of Y. Since $f(x) \varepsilon 2^Y$, $f(x) \neq \square$ for any $x \varepsilon X$, so $xR = f(x) \neq \square$ for any $x \varepsilon X$. Hence $X = RY$, so $R \varepsilon S(X,Y)$. By definition R is the graph of f in X x Y, and it is clearly unique.

Define $i\colon (2^Y)^X \longrightarrow S(X,Y)$ by letting $i(f)$ equal the graph of f in X x Y. In view of 1.1, i is 1-1 and onto.

Let $\mathcal{C}$ be the compact open topology for $(2^Y)^X$: that is, $\mathcal{C}$ is generated by $\{B(K,W) \mid K$ compact in X and W open in $2^Y\}$. Let $\mathcal{R}$ be the topology for $S(X,Y)$ such that i is a homeomorphism onto $(S(X,Y), \mathcal{R})$. We will call $\mathcal{R}$ the <u>compact open topology for the relation space $S(X,Y)$</u>.

An obvious problem is how to describe $\mathcal{N}$ directly, and 1.2 gives such a description, though it is not very satisfactory. There may be a simpler description for $\mathcal{N}$ which we have overlooked. Define

$$\Delta(K,U) = \{R \in S(X,Y) \mid KR \subset U\} \text{ and}$$

$$\Sigma(K,U) = \{R \in S(X,Y) \mid K \subset RU\}.$$

Let $M(x,U)$ denote a set of type $\Delta(x,U)$ or $\Sigma(x,U)$, and let $\mathcal{P}$ be the collection of all sets of the form

$$\bigcap_{x \in K} \{ \bigcup_{a \in D} [\bigcap_{i=1}^{n_a} M(x,U_{ai})]\}$$

where K is compact in X, each U_{ai} is open in Y, and D is any indexing set.

(1.2) <u>Theorem</u>. $\mathcal{N}$, the compact open topology for $S(X,Y)$, is generated by $\mathcal{P}$.

Proof: Let $\mathcal{C}$ be the compact open topology for $(2^Y)^X$, let $B(K,W)$ be an arbitrary subbase element of $((2^Y)^X, \mathcal{C})$, and let $i: (2^Y)^X \longrightarrow S(X,Y)$ be the function identifying each f with its graph in $X \times Y$. By definition of $\mathcal{N}$, $i(B(K,W))$ is an arbitrary subbase element of $(S(X,Y), \mathcal{N})$.

Since W is an arbitrary open set of 2^Y, $W = \bigcup_{a \in D} N_a$, where each N_a is a finite intersection of subbase elements of 2^Y. The following equalities are easily established:

$$B(K, \bigcup_{a \varepsilon D} N_a) = \bigcap_{x \varepsilon K} B(x, \bigcup_{a \varepsilon D} N_a)$$

$$= \bigcap_{x \varepsilon K} \bigcup_{a \varepsilon D} B(x, N_a).$$

Fix a in D; then $N_a = \bigcap_{j=1}^{m} V_j$, where each V_j is of the form 2^{U_j} or 2_{U_j} for some U_j open in Y. The following equalities are easily established:

$$B(x, \bigcap_{j=1}^{m} V_j) = \bigcap_{j=1}^{m} B(x, V_j),$$

$$i(B(x, 2^U)) = \Delta(x, U), \text{ and}$$

$$i(B(x, 2_U)) = \Sigma(x, U).$$

Finally, since i is 1-1 it can be distributed through intersections, so

$$i(B(K,W)) = \bigcap_{x \varepsilon K} \bigcup_{a \varepsilon D} \bigcap_{j_a = 1}^{m_a} i(B(x, V_{j_a}))$$

$$= \bigcap_{x \varepsilon K} \bigcup_{a \varepsilon D} \bigcap_{j_a = 1}^{m_a} M(x, U_{j_a})$$

and this last set is in $\mathscr{P}$ by definition. This completes the proof.

Let $\mathscr{B}$ be the topology for $(2^Y)^X$ generated by $\{B(K, 2^U)$ and $B(K, 2_U) \mid K$ compact in X and U open in Y$\}$. We will call $\mathscr{B}$ the <u>subcompact open topology for $(2^Y)^X$</u>, and the topology for S(X,Y) which is homeomorphic to $\mathscr{B}$ will be called the <u>subcompact open topology for S(X,Y)</u>.

Let $\mathcal{M}$ be the topology for $S(X,Y)$ generated by $\{\Delta(K,U)$ and $\Sigma(K,U) \mid K$ compact in X and U open in $Y\}$.

(1.3) Theorem. $\mathcal{M}$ is the subcompact open topology for $S(X,Y)$, and the compact open topology for $S(X,Y)$, $\mathcal{N}$, contains $\mathcal{M}$.

Proof: Let i be the function of 1.2. Then by definition the subcompact open topology for $S(X,Y)$ is generated by $\{i(B(K,2^U))$ and $i(B(K,2_U)) \mid K$ compact in X and U open in $Y\}$; $i(B(K,2^U)) = \Delta(K,U)$ and $i(B(K,2_U)) = \Sigma(K,U)$, which completes the proof that $\mathcal{M}$ is the subcompact open topology for $S(X,Y)$.

Since $B(K,2^U)$ and $B(K,2_U)$ are in the compact open topology for $(2^Y)^X$, their images under i will be in $\mathcal{N}$ by definition. This completes the proof that $\mathcal{N} \supset \mathcal{M}$.

If X is a space whose only compact sets are finite, then of course $\mathcal{M} \supset \mathcal{P}$, hence $\mathcal{N} = \mathcal{M}$. However, in general $\mathcal{N}$ properly contains $\mathcal{M}$, as the following example shows.

(1.4) Example. Let I be the unit interval, and define $R \subset I \times I$ as follows:

$$xR = \begin{cases} 0 \text{ if } x < \frac{1}{2} \\ 1 \text{ if } x \geq \frac{1}{2}. \end{cases}$$

Let $U = [0,\frac{1}{4})$ and $V = (\frac{3}{4},1]$; note $B(I,2^U \cup 2^V)$ is open in $((2^I)^I, \mathcal{C})$ and $R \in i(B(I,2^U \cup 2^V))$, which is open in $(S(I,I),\mathcal{N})$ by definition of $\mathcal{N}$ (i is the function used to define $\mathcal{N}$).

An arbitrary basic set of $\mathscr{M}$ is

$$M = \left(\bigcap_{i=1}^{n} \Delta(K_i, U_i)\right) \cap \left(\bigcap_{j=1}^{m} \Sigma(L_j, V_j)\right)$$

where each K_i and L_j is compact in I and each U_i and V_j is open in I. Suppose $R \in M$. We will prove $M \not\subset i(B(I, 2^U \cup 2^V))$.

Case 1. $\frac{1}{2} \notin \bigcup_{i=1}^{n} K_i$. Then there is some $\varepsilon > 0$ such that the open interval $(\frac{1}{2} - \varepsilon, \frac{1}{2}) \subset I \setminus \bigcup_{i=1}^{n} K_i$. Let $x_0 = \frac{1}{2} - \frac{\varepsilon}{2}$, and define $S \subset I \times I$ thus:

$$xS = \begin{cases} xR \text{ if } x \neq x_0 \\ (x_0)R \cup 1 \text{ if } x = x_0. \end{cases}$$

It is easily seen that $S \in M$ but $S \notin i(B(I, 2^U \cup 2^V))$ since if $f_S = i^{-1}(S)$, $f_S(x_0) \notin 2^U \cup 2^V$ so $f_S \notin B(I, 2^U \cup 2^V)$.

Case 2. $\frac{1}{2} \in \bigcup_{i=1}^{n} K_i$. Then we may suppose $\frac{1}{2} \in \bigcap_{i=1}^{p} K_i$ and $\frac{1}{2} \notin \bigcup_{i=p+1}^{n} K_i$. Again there is some $\varepsilon > 0$ such that $(\frac{1}{2} - \varepsilon, \frac{1}{2}) \subset I \setminus \bigcup_{i=p+1}^{n} K_i$. Let $x_0 = \frac{1}{2} - \frac{\varepsilon}{2}$ and define S as before. Again it is easily seen that $S \in M$ but $S \notin i(B(I, 2^U \cup 2^V))$ since $f_S(x_0) \notin 2^U \cup 2^V$.

The following lemma is an easy consequence of the fact that a compact Hausdorff space is regular. It appears in [9].

(1.5) Lemma. If C is compact in a Hausdorff space X and if $\{U_i\}_{i=1}^{n}$ is a finite open cover of C, then there are closed sets $\{C_i\}_{i=1}^{n}$ such that $C_i \subset U_i$ for each i and $C = \bigcup_{i=1}^{n} C_i$.

Proof: $\{U_i \cap C\}_{i=1}^{n}$ is a collection of sets open in C. C is regular so if $x \in U_i \cap C$, there is some V_x open in C such that $v \in V_x \subset V_x^* \subset U_i \cap C$. $\{V_x \mid x \in C\}$ is an open cover of C, so there is a finite subcover, $\{V_{x_j}\}_{j=1}^{m}$. Let $C_i = \cup\{V_{x_j}^* \mid V_{x_j}^* \subset U_i \cap C\}$, for each i. Then C_i is a finite union of closed sets, hence is closed; $C_i \subset U_i \cap C \subset U_i$; and finally, each V_{x_j} was chosen so that $V_{x_j}^* \subset U_i \cap C$ for some i, so $\bigcup_{i=1}^{n} C_i = \bigcup_{j=1}^{m} V_{x_j}^* = C$.

If $\mathcal{C}$ is a topology for a space S and $T \subset S$, $\underline{\mathcal{C} \mid T}$ will denote $\{C \cap T \mid C \in \mathcal{C}\}$.

The following lemma is known.

(1.6) Lemma. Let Z be Hausdorff, X a space and $\mathcal{C}$ be the compact open topology for Z^X. Let $T = \{f\colon X \longrightarrow Z \mid f$ is continuous on every compact subset of $X\}$ and let $\mathcal{V}$ be a subbasis for Z. Then $\mathcal{C} \mid T$ is generated by $\{B(K,V) \cap T \mid K$ compact in X and $V \in \mathcal{V}\}$.

Proof: Let K be any compact set in X and W any open set of Z. Then $W = \bigcup_{a \in D} U_a$, where each U_a is a finite intersection of elements

of $\mathcal{V}$. By definition of the compact open topology, $B(K,W)$ is an arbitrary subbase element of $\mathcal{E}$. Let $f \in B(K,W) \cap T$, which implies $f(K) \subset \bigcup_{a \in D} U_a$. f in T implies f continuous on K, so $f(K)$ is compact; hence there is a finite collection, $\{U_{ai}\}_{i=1}^{n}$, which covers $f(K)$. $f(K) \subset \bigcup_{i=1}^{n} U_{ai}$ and $f(K)$ compact in the Hausdorff space Z, so by 1.3 there are closed sets C_i such that $f(K) = \bigcup_{i=1}^{n} C_i$ and $C_i \subset U_{ai}$ for each $i = 1,..,n$. Let $K_i = K \cap f^{-1}(C_i)$ for each i and note that f continuous on K and K compact imply that each K_i is compact. Note also that $f(K_i) \subset C_i \subset U_{ai}$ for each i. Then $\bigcap_{i=1}^{n} B(K_i, U_{ai}) \in \mathcal{E}$ and $f \in \bigcap_{i=1}^{n} B(K_i, U_{ai}) \cap T$, which is clearly contained in $B(K,W) \cap T$.

To complete the proof, recall that for each U_{ai}, $U_{ai} = \bigcap_{j=1}^{m} V_j$ for some finite subcollection of $\mathcal{V}$, $\{V_j\}_{j=1}^{m}$, and note that $B(L, \bigcap_{j=1}^{m} V_j) = \bigcap_{j=1}^{m} B(L, V_j)$ for any sets L and V_j.

Let Z be a set and $U \subset Z$. We define $\underline{\overline{2}}^Z$ to be the collection of all nonnull closed subsets of Z, and $\underline{\overline{2}_U}$ to be the collection of all nonnull closed subsets of Z which have a nonnull intersection with U.

The following lemma is easily proved. It can be found in [11], [13] and [15].

(1.7) <u>Lemma</u>. If Y is regular and if $\overline{2}^Y$ has the topology generated by $\{\overline{2}^U$ and $\overline{2}_U \mid U$ open in $Y\}$, then $\overline{2}^Y$ is Hausdorff.

(1.6') <u>Corollary</u>. If X is a space, Y is regular, $\mathscr{C}$ is the compact open topology for $(\overline{2}^Y)^X$, and if $T = \{f\colon X \longrightarrow \overline{2}^Y \mid f$ is continuous on the compact sets of $X\}$, then $\mathscr{C} \mid T$ is generated by $\{B(K,\overline{2}^U) \cap T$ and $B(K,\overline{2}_U) \cap T \mid K$ compact in X and U open in $Y\}$.

Let X and Y be spaces, and let $R \subset X \times Y$. R is said to be <u>upper semicontinuous on X</u> iff for every x in X and neighborhood V of xR, there is a neighborhood U of x such that $UR \subset V$. We will use <u>u.s.c.</u> to abbreviate upper semicontinuous. Notice that R is u.s.c. on X iff for each V open in Y, $\{x \in X \mid xR \subset V\}$ is open in X.

R is said to be <u>lower semicontinuous on X</u> iff for every V open in Y, RV is open in X. We will use <u>l.s.c.</u> to abbreviate lower semicontinuous. Notice that R is l.s.c. on X iff for each V open in Y, $\{x \in X \mid xR \cap V \neq \square\}$ is open in X.

These names are those used by Kuratowski in [10] and by other authors. In [15] and other papers, Strother called these forms of continuity <u>weak continuity</u> and <u>residual continuity</u>, respectively.

The following remark will be useful in Section 4 of Part II. If $K \subset X$, $R \subset K \times Y$ and R is u.s.c. and l.s.c. on K, then $R \subset X \times Y$ but R need not be u.s.c. or l.s.c. on X. First consider upper semicontinuity: let V be open in Y and $U = \{x \in K \mid xR \subset V\}$; U is open in K, but $U \cup (X \setminus K) = \{x \in X \mid xR \subset V\}$ need not be open in X.

Similarly for lower semicontinuity: if V is open in Y, since $R \subset K \times Y$, $RV \subset K$; RV is open in K but need not be open in X.

If A and B are subsets of a set X, $A \setminus B$ will denote $\{a \in A \mid a \notin B\}$.

The following lemma is due to many authors.

(1.8) <u>Lemma</u>. If $f\colon X \longrightarrow 2^Y$ is continuous, the graph of f in $X \times Y$ is u.s.c. and l.s.c. on X. Conversely, if $R \in S(X,Y)$ and R is u.s.c. and l.s.c. on X, then the function $f\colon X \longrightarrow 2^Y$ whose graph in $X \times Y$ is R is continuous.

Proof: Let $f\colon X \longrightarrow 2^Y$ be continuous and let R be the graph of f in $X \times Y$. Then for any U open in Y, $f^{-1}(2^U) = \{x \in X \mid xR \subset U\}$ is open in X and $f^{-1}(2_U) = RU$ is open in X, hence R is u.s.c. and l.s.c. on X.

Conversely, if $R \in S(X,Y)$ and R is u.s.c. and l.s.c. on X, define $f\colon X \longrightarrow 2^Y$ by $f(x) = xR$ for each x in X. f is continuous if $f^{-1}(2_U)$ and $f^{-1}(2^U)$ are open for arbitrary U open in Y; R u.s.c. on X implies $\{x \in X \mid xR \subset U\} = f^{-1}(2^U)$ is open and R l.s.c. on X implies $RU = f^{-1}(2_U)$ is open. This completes the proof.

If $R \subset X \times Y$, we will say R is <u>point closed</u> iff xR is closed for each x in X.

Recall that $\Delta(K,U) = \{R \subset X \times Y \mid X = RY \text{ and } KR \subset U\}$ and $\Sigma(K,U) = \{R \subset X \times Y \mid X = RY \text{ and } K \subset RU\}$.

Since $\mathfrak{R}$ is difficult to describe directly, the following lemma is very useful.

(1.9) Theorem. Let X be a space, Y be regular and $F = \{R \subset X \times Y \mid X = RY$, R is point closed, and for each compact $K \subset X$, $R \cap (K \times Y)$ is u.s.c. and l.s.c. on $K\}$. Then $\mathcal{N} \mid F$, the compact open topology for F, equals $\mathcal{M} \mid F$, the subcompact open topology for F: that is, $\mathcal{N} \mid F$ is generated by $\{\Delta(K,U) \cap F$ and $\Sigma(K,U) \cap F \mid K$ compact in X and U open in $Y\}$.

Proof: Let $i\colon (2^Y)^X \longrightarrow S(X,Y)$ be the function of 1.2, and let $T = \{f\colon X \longrightarrow \overline{2}^{\,Y} \mid f$ is continuous on each compact $K \subset X\}$. By 1.8, $i(T) = F$, and if $\mathcal{C}$ is the compact open topology for $(2^Y)^X$, by definition of $\mathcal{N}$, $i\colon (T, \mathcal{C} \mid T) \longrightarrow (F, \mathcal{N} \mid F)$ is a homeomorphism.

By 1.6', $\mathcal{C} \mid T$ is generated by $\{B(K,\overline{2}^{\,U}) \cap T$ and $B(K,\overline{2}_U) \cap T \mid K$ compact in X and U open in $Y\}$. It is easy to see that $i(B(K,\overline{2}^{\,U})) = \Delta(K,U)$ and $i(B(K,\overline{2}_U)) = \Sigma(K,U)$, which completes the proof.

Let us recall several things.

Following Kelley in [8], if $\mathcal{J}$ is a topology for Z^X and $T \subset Z^X$, we define $\mathcal{J} \mid T$ to be jointly continuous on K iff $K \subset X$ and the function $p\colon T \times K \longrightarrow Z$ defined by $p(f,x) = f(x)$ is continuous.

$\mathcal{C}$, the compact open topology for Z^X, is characterized by these facts, proofs of which can be found in Chapter 7 of [8]:

(1.10) Theorem. If $\mathcal{J}$ is a topology for Z^X which is jointly continuous on compacta, $\mathcal{J}$ is larger than $\mathcal{C}$.

(1.11) <u>Theorem</u>. When X is Hausdorff or regular and T is a subset of Z^X containing only functions continuous on the compacta of X, $\mathcal{C} \mid T$ is jointly continuous on each compact $K \subset X$.

Letting $Z = 2^Y$ and drawing the obvious parallels with S(X,Y), we define an evaluation relation $Q \subset ((S(X,Y) \times X) \times Y)$ thus: $(R,x)Q = xR$ for each $(R,x) \varepsilon S(X,Y) \times X$; and if $\mathcal{S}$ is a topology for S(X,Y) and $F \subset S(X,Y)$, we define $\mathcal{S} \mid F$ to be <u>jointly continuous on K</u> iff $K \subset X$ and $Q \cap ((F \times K) \times Y)$ is u.s.c. and l.s.c. on $F \times K$.

These definitions will give theorems for S(X,Y), 1.10' and 1.11', which parallel 1.10 and 1.11. The following lemma is useful in the proofs of 1.10' and 1.11'.

(1.12) <u>Lemma</u>. Let S(X,Y) have topology $\mathcal{S}$ and $(2^Y)^X$ have topology $\mathcal{J}$. Let $T \subset (2^Y)^X$ and let $F = i(T)$, where $i(f)$ is the graph of f in $X \times Y$, for each f in T. Then if $K \subset X$ and if $i\colon (T, \mathcal{J} \mid T) \longrightarrow (F, \mathcal{S} \mid F)$ is a homeomorphism, $\mathcal{J} \mid T$ is jointly continuous on K iff $\mathcal{S} \mid F$ is jointly continuous on K.

Proof: We need to prove $Q' = Q \cap ((F \times K) \times Y)$ u.s.c. and l.s.c. on $F \times K$ iff $p\colon T \times K \longrightarrow 2^Y$ is continuous. By 1.7, Q' is u.s.c. and l.s.c. on $F \times K$ iff $q\colon F \times K \longrightarrow 2^Y$ is continuous, where $q(R,x) = (R,x)Q' = xR$ for each $(R,x) \varepsilon F \times K$. Since $p(f,x) = q(i(f),x)$ and i is a homeomorphism, it is clear that p is continuous iff q is. This completes the proof.

(1.10') Theorem. If $\mathcal{S}$ is a topology for $S(X,Y)$ which is jointly continuous on compacta, $\mathcal{S}$ is larger than $\mathcal{N}$, the compact open topology for $S(X,Y)$.

Proof: Let $i: (2^Y)^X \longrightarrow S(X,Y)$ be the function such that $i(f)$ is the graph of f in $X \times Y$, for each $f \varepsilon (2^Y)^X$, and let $\mathcal{J}$ be the topology for $(2^Y)^X$ such that i is a homeomorphism onto $(S(X,Y),\mathcal{S})$. By hypothesis, if K is compact in X, $Q \cap ((S(X,Y) \times K) \times Y)$ is u.s.c. and l.s.c. on $S(X,Y) \times K$; then 1.12 implies that $p: (2^Y)^X \times K \longrightarrow 2^Y$ is continuous, when $(2^Y)^X$ has topology $\mathcal{J}$. Hence $\mathcal{J}$ is jointly continuous on K, which is an arbitrary compact set, so by 1.10, $\mathcal{J} \supset \mathcal{C}$. Then by the homeomorphisms which define $\mathcal{N}$ and $\mathcal{S}$, $\mathcal{S} \supset \mathcal{N}$.

(1.11') Theorem. When X is Hausdorff or regular and F is a subset of $S(X,Y)$ containing only relations R such that for each compact $K \subset X$, $R \cap (K \times Y)$ is u.s.c. and l.s.c. on K, then $\mathcal{N} \mid F$, the compact open topology for F, is jointly continuous on each compact $K \subset X$.

Proof: Let $i: (2^Y)^X \longrightarrow S(X,Y)$ be the function of 1.10' and let $T = i^{-1}(F)$. Then if $\mathcal{C}$ is the compact open topology for $(2^Y)^X$, by definition of $\mathcal{N}$, $i: (T, \mathcal{C} \mid T) \longrightarrow (F, \mathcal{N} \mid F)$ is a homeomorphism.

By 1.8 and hypothesis, each f in T is continuous on each compact $K \subset X$; and then by 1.11, $\mathcal{C} \mid T$ is jointly continuous on compacta. Therefore, by 1.12, $\mathcal{N} \mid F$ is jointly continuous on compacta.

SECTION 3. A relation space on which the compact open topology is metrizable.

1.13 and the following definitions and remark are due to Arens. (See [1].)

A space X will be called <u>hemicompact</u> iff X is the union of a denumerable collection of compact sets, $\{H_i\}_{i=1}^{n}$, and for any compact $K \subset X$, there is some $N_K < \infty$ such that $K \subset \bigcup_{i=1}^{N_K} H_i$. This is a true generalization of compactness, since for example any locally compact space with a denumerable basis is hemicompact.

(1.13) <u>Theorem</u>. Let X be hemicompact, Z be a metric space, and let $T = \{f\colon X \longrightarrow Z \mid f \text{ is continuous}\}$. Then the compact open topology for T is metrizable.

(1.14) <u>Theorem</u>. Let X be hemicompact, let (Y,ρ) be a metric space, and let $F = \{R \subset X \times Y \mid X = RY,\ xR \text{ is closed and bounded for each } x \text{ in } X, \text{ and } R \text{ is u.s.c. and l.s.c.}\}$. If $\mathcal{R}$ is the compact open topology for $S(X,Y)$, $\mathcal{R} \mid F$ is metrizable.

Proof: Let $V_r(A) = \{y \in Y \mid \rho(y,A) < r\}$ for any $A \subset Y$ and real r; let $Z = \{A \subset Y \mid A \text{ is closed and bounded}\}$; and define $d\colon Z \times Z \longrightarrow \underline{R}$ thus: $d(A,B) = \inf \{r \in \underline{R} \mid A \subset V_r(B) \text{ and } B \subset V_r(A)\}$. For a proof that d is a metric for Z, see p. 167 in [6]. d is known as the Hausdorff metric.

Let $T = \{f\colon X \longrightarrow Z \mid f \text{ is continuous}\}$; then if $i\colon T \longrightarrow S(X,Y)$

maps each f in T onto its graph in X x Y, which is in S(X,Y) by 1.1, i(T) = F by 1.8. Let $\mathscr{C}$ be the compact open topology for $(2^Y)^X$ and note $Z \subset 2^Y$ implies $T \subset (2^Y)^X$, so $\mathscr{C} \mid T$ is the compact open topology for T. Z is metric, so by 1.13, $\mathscr{C} \mid T$ is metrizable.

By definition of $\mathscr{N}$, i: $(T, \mathscr{C} \mid T) \longrightarrow (F, \mathscr{N} \mid F)$ is a homeomorphism; therefore, since $\mathscr{C} \mid T$ is metrizable, $\mathscr{N} \mid F$ is also.

(1.14') Corollary. If $\mathscr{M}$ is the subcompact open topology for S(X,Y), $\mathscr{M} \mid F$ is metrizable.

Proof: By definition of F and by 1.9, $\mathscr{N} \mid F = \mathscr{M} \mid F$.

SECTION 4. A relation space with the compact open topology which is a semialgebra.

Let (Y,o) be a semigroup and let Y have topology $\mathcal{J}$. Y is said to be a <u>topological semigroup</u> iff the function $m\colon Y \times Y \longrightarrow Y$ defined by $m(y,y') = y \circ y'$ is continuous, when $Y \times Y$ has the usual product topology.

If A and B are subsets of a semigroup (Y,o), we define <u>$A \circ B$</u> to be $\{a \circ b \mid a \in A \text{ and } b \in B\}$ Recall that $S(X,Y) = \{R \subset X \times Y \mid X = RY\}$, and hence if Q and R are in $S(X,Y)$ and $x \in X$, neither xQ nor xR is null, so $(xQ) \circ (xR) \neq \square$. Let us define <u>$Q \circ R$</u> $= \cup \{\{x\} \times ((xQ) \circ (xR)) \mid x \in X\}$, and note $Q \circ R \subset X \times Y$ and $Q \circ R \in S(X,Y)$. Notice also that for any $K \subset X$, $K(Q \circ R) = (KQ) \circ (KR)$. We state these things formally, for reference purposes.

(1.15) <u>Lemma</u>. If X is a set and (Y,o) is a semigroup, then $(S(X,Y),o)$ is also a semigroup. If $K \subset X$ and $Q,R \in S(X,Y)$, then $K(Q \circ R) = (KQ) \circ (KR)$.

It is not possible to have $S(X,Y)$ a group even if Y is a group; for suppose Y is a nondegenerate group with identity $\emptyset$, and let $P = X \times \{\emptyset\}$. This is the only possible identity for any relation which is single-valued at any point, and it is obvious how to define an inverse for any relation $R \in S(X,Y)$ which is the graph of a single-valued function from X to Y. However, if $R \in S(X,Y)$ and there is some x in X such that xR contains more than one point, then there is no

$Q \in S(X,Y)$ such that $RQ = P$. In other words, the only relations which have inverses are those relations which are graphs of single-valued functions.

In 1.16 and 1.18, let the operation on the topological semigroup be o, and let the operation induced in $S(X,Y)$ be o, as defined above.

If Y is a space and $R \subset X \times Y$, we define R to be <u>point compact</u> iff xR is compact for each x in X. Let us define <u>$C(X,Y)$</u> to be $\{R \subset X \times Y \mid X = RY, R$ is u.s.c. and l.s.c. on X and point compact$\}$.

The following lemma appears in [4].

(1.16) <u>Lemma</u>. If X is a space, Y is a topological semigroup, $S(X,Y) = \{R \subset X \times Y \mid X = RY\}$ and $C(X,Y) = \{R \subset X \times Y \mid X = RY, R$ is u.s.c. and l.s.c. on X and point compact$\}$, then $C(X,Y)$ is a subsemigroup of $S(X,Y)$. That is, if $Q,R \in C(X,Y)$, $Q \circ R \in C(X,Y)$ also.

Proof: Let $m\colon Y \times Y \longrightarrow Y$ be the continuous operation of Y, let Q and R be any elements of $C(X,Y)$, and let U be any open set of Y.

(i) $Q \circ R$ is u.s.c. Let $x \in X$ and $x(Q \circ R) \subset U$; that is $(xQ) \circ (xR) \subset U$. $m^{-1}(U)$ is open in $Y \times Y$ and $(xQ) \times (xR) \subset m^{-1}(U)$; xQ and xR compact so by a theorem of Wallace (see [8], p. 142), there are open sets V and W in Y such that $(xQ) \times (xR) \subset V \times W \subset m^{-1}(U)$. Q and R are u.s.c., so there are open sets V' and W' in X such that $x \in V' \cap W'$ and $V'Q \subset V$, $W'R \subset W$. Let $O = V' \cap W'$. Then $x \in O$, O is open, and $O(Q \circ R) = (OQ) \circ (OR) \subset V \circ W = m(V \times W) \subset U$. Therefore $Q \circ R$ is u.s.c.

(ii) $Q \circ R$ is l.s.c. Let $x \in (Q \circ R)U$ and $y \in x(Q \circ R) \cap U$. $y \in (xQ) \circ (xR)$ implies there are $y_1 \in xQ$ and $y_2 \in xR$ such that $y_1 \circ y_2 = y$; $y \in U$ implies $(y_1, y_2) \in m^{-1}(U)$, which is open. So there are open sets V and W in Y such that $(y_1, y_2) \in V \times W \quad m^{-1}(U)$. Q and R are l.s.c., so QV and RW are open in X. Note $x \in O = QV \cap RW$, which is open; and further, $O \subset (Q \circ R)U$: for if $x' \in O$, there exist $y_1' \in x'Q \cap V$ and $y_2' \in x'R \cap W$, so $(y_1', y_2') \subset V \times W$, which implies $y_1' \circ y_2' \subset U$. That is, $y_1' \circ y_2' \in x'(Q \circ R) \cap U$. So $Q \circ R$ is l.s.c.

(iii) For each $x \in X$, $x(Q \circ R)$ is compact. $x(Q \circ R) = (xQ) \circ (xR) = m((xQ) \times (xR))$, which is compact since m is continuous and xQ and xR are compact.

The following lemma is Theorem 3.2 of [13].

(1.17) Lemma. Let I be compact in a space X, let Y be a space and let $R \subset X \times Y$ be u.s.c. on X and point compact. Then IR is compact.

Proof: Let $\mathcal{U}$ be a collection of sets open in Y which covers IR. For each x in I, xR is compact and is covered by $\mathcal{U}$, so we can select a finite subcover, $\mathcal{U}_x$. Then $xR \subset \cup \mathcal{U}_x$, open in Y; R is u.s.c. on X so there is an open set V_x in X such that $x \in V_x$ and $V_x R \subset \cup \mathcal{U}_x$. Find such V_x for each x in I, and select a finite subcover of I, $\{V_{xi}\}_{i=1}^{n}$. Then $\cup \{\mathcal{U}_{xi}\}_{i=1}^{n}$ is a finite subcollection of $\mathcal{U}$ and covers IR.

(1.18) Theorem. Let X be a Hausdorff space, Y a topological semigroup which is regular, and let $\mathcal{R}$ be the compact open topology for $S(X,Y) = \{R \subset X \times Y \mid X = RY\}$. Then $(C(X,Y), \mathcal{R} \mid C(X,Y))$ is a topological semigroup, where $C(X,Y) = \{R \subset X \times Y \mid X = RY$, R is u.s.c. and l.s.c. on X and point compact$\}$.

Proof: For brevity we will let C denote $C(X,Y)$ in this proof. By 1.16, C is a semigroup under the operation o, so we need to prove that $f: C \times C \longrightarrow C$ defined by $f(Q,R) = Q \circ R$ is continuous, when $C \times C$ has the usual product topology. It will suffice to prove that the inverse of each member of a subbase for C is open, and by 1.9, since Y is regular, $\{\Delta(K,U) \cap C$ and $\Sigma(K,U) \cap C \mid K$ is compact in X and U is open in $Y\}$ is a subbase for $\mathcal{R} \mid C$.

(i) Let $(Q,R) \in f^{-1}(\Delta(K,U) \cap C)$. We will find an open set about (Q,R) which lies inside $f^{-1}(\Delta(K,U) \cap C)$. First note that $f(Q,R) = Q \circ R \in \Delta(K,U)$ implies $K(Q \circ R) \subset U$, or, by 1.15, $(KQ) \circ (KR) \subset U$. Since K is compact and Q and R are in C, by 1.17, KQ and KR are compact. $(KQ) \times (KR) \subset m^{-1}(U)$, which is open since m is continuous, so by a theorem of Wallace (see p. 142 in [8]), there are open sets V and W in Y such that $KQ \subset V$, $KR \subset W$, and $m(V \times W) = V \circ W \subset U$. Then $Q \in \Delta(K,V) \cap C$, $R \in \Delta(K,W) \cap C$, and both these sets are open in C. Finally, if $(Q',R') \in (\Delta(K,V) \cap C) \times (\Delta(K,W) \cap C)$, $f(Q',R') \in \Delta(K,U) \cap C$: for $KQ' \subset V$ and $KR' \subset W$ implies $K(Q' \circ R') = (KQ') \circ (KR') \subset V \circ W \subset U$, and $f(Q',R') = Q' \circ R'$.

(ii) Let $(Q,R) \varepsilon f^{-1}(\Sigma(K,U) \cap C)$. We will find an open set about (Q,R) which lies inside $f^{-1}(\Sigma(K,U) \cap C)$. First note that $f(Q,R) = Q \circ R \varepsilon \Sigma(K,U)$ implies $K \subset (Q \circ R)U$, so for each x in K, there is some $y \varepsilon x(Q \circ R) \cap U = (xQ \circ xR) \cap U$. Find $y_1 \varepsilon xQ$ and $y_2 \varepsilon xR$ such that $y_1 \circ y_2 = y$, and then use the continuity of m to find V and W open in Y such that $y_1 \varepsilon V$, $y_2 \varepsilon W$ and $V \circ W \subset U$. Find such V_x and W_x for each x in K; note $x \varepsilon QV_x \cap RW_x$ and this is open in X since Q and R are l.s.c. on X, for each x in K, so the collection $\{QV_x \cap RW_x \mid x \varepsilon K\}$ is an open cover of K. K is compact so there is a finite subcover, $\{QV_i \cap RW_i\}_{i=1}^{n}$. Since X is Hausdorff, by 1.5 there are closed sets K_i such that $K = \bigcup_{i=1}^{n} K_i$ and $K_i \subset QV_i \cap RW_i$ for each i.

Let $\underline{V} = \bigcap_{i=1}^{n} \Sigma(K_i,V_i)$ and $\underline{W} = \bigcap_{i=1}^{n} \Sigma(K_i,W_i)$. It is clear that $(Q,R) \varepsilon (\underline{V} \cap C) \times (\underline{W} \cap C)$, and that this set is open in $C(X,Y)$. Finally, if $(Q',R') \varepsilon (\underline{V} \cap C) \times (\underline{W} \cap C)$, $Q' \circ R' \varepsilon C$ by 1.16; if $x \varepsilon K$ there is some i such that $x \varepsilon K_i$; $Q' \varepsilon \Sigma(K_i,V_i)$ and $R' \varepsilon \Sigma(K_i,W_i)$, hence there is $y_1 \varepsilon xQ' \cap V_i$ and $y_2 \varepsilon xR' \cap W_i$ and then $(y_1 \circ y_2) \varepsilon x(Q' \circ R') \cap (V_i \circ W_i)$. Therefore $(y_1 \circ y_2) \varepsilon x(Q' \circ R') \cap U$, which says $x \varepsilon (Q' \circ R')U$. x was arbitrary in K, hence $Q' \circ R' \varepsilon \Sigma(K,U)$, and this completes the proof.

Let Y be a topological semigroup, let $A \subset Y$ and let $p \varepsilon Y$. Then <u>$p \circ A$</u> is defined to be $\bigcup \{p \circ a \mid a \varepsilon A\}$. If X is a set and $Q \subset X \times Y$, <u>$p \circ Q$</u> is defined to be $\bigcup \{\{x\} \times (p \circ (xQ)) \mid x \varepsilon X\}$, and $p \circ Q$ is called a <u>scalar o-multiple</u> of Q, or just <u>scalar multiple</u> of Q. Notice that p is not a relation in X x Y, but $p \circ Q$ is.

(1.19) <u>Lemma</u>. Let X be a space, Y a topological semigroup, and $C(X,Y) = \{R \subset X \times Y \mid X = RY, R$ is u.s.c. and l.s.c. on X and point compact$\}$. Then $C(X,Y)$ is closed under scalar multiplication.

Proof: Let $Q \varepsilon C(X,Y)$ and $p \varepsilon Y$. Define $P \subset X \times Y$ thus: $xP = p$ for each x in X. It is easy to see that $P \varepsilon C(X,Y)$, and then by 1.16, $P \circ Q \varepsilon C(X,Y)$. Finally, $p \circ Q = P \circ Q$: for let $x \varepsilon X$; then $x(p \circ Q) = p \circ (xQ)$ by definition of scalar multiplication, and $p = xP$, so $x(p \circ Q) = (xP) \circ (xQ)$, which is $x(P \circ Q)$ by definition. Therefore $p \circ Q \varepsilon C(X,Y)$. This completes the proof.

Let $\underline{R}$ be the real numbers, let X be a space and let $C \subset S(X,\underline{R}) = \{P \subset X \times \underline{R} \mid X = P\underline{R}\}$. We will call C a <u>semialgebra</u> iff

(i) C is a topological semigroup under both addition and multiplication; and

(ii) C is closed under scalar multiplication.

(1.20) <u>Theorem</u>. Let X be Hausdorff, $\underline{R}$ the real numbers, and let $\mathcal{R}$ be the compact open topology for $S(X,\underline{R}) = \{P \subset X \times \underline{R} \mid X = P\underline{R}\}$. Let $C(X,\underline{R}) = \{P \subset X \times \underline{R} \mid X = P\underline{R}, P$ is u.s.c. and l.s.c. on X and point compact$\}$. Then $(C(X,\underline{R}), \mathcal{R} \mid C(X,\underline{R}))$ is a semialgebra.

Proof: (i) It is well known that $\underline{R}$ is a topological semigroup under both addition and multiplication; so by 1.18, $C(X,\underline{R})$ is also a topological semigroup under addition and multiplication.

(ii) Since $\underline{R}$ is a topological semigroup under multiplication, by 1.19, $C(X,\underline{R})$ is closed under scalar multiplication.

(1.20') Corollary. Let X be Hausdorff, $\underline{R}$ the real numbers, and let $\mathscr{M}$ be the subcompact open topology for $S(X,\underline{R})$. Then $(C(X,\underline{R}), \mathscr{M} \mid C(X,\underline{R}))$ is a semialgebra.

Proof: $C(X,\underline{R})$ contains only relations which are point closed and u.s.c. and l.s.c. on X. Certainly $\underline{R}$ is regular, hence by 1.9, $\mathscr{R} \mid C(X,\underline{R}) = \mathscr{M} \mid C(X,\underline{R})$. Then by 1.20, $(C(X,\underline{R}), \mathscr{M} \mid C(X,\underline{R}))$ is a semialgebra.

SECTION 5. An isomorphism theorem for semialgebras of relations.

Let $\underline{R}$ be the real numbers and, for any space X, let $C(X) = \{h: X \longrightarrow \underline{R} \mid h \text{ is continuous}\}$. It is well known that, when we define $(f + h)(x) = f(x) + h(x)$ and $(fh)(x) = f(x)h(x)$, $C(X)$ is a ring, is closed under scalar multiplication, and, with respect to this algebra on $C(X)$, the following theorem is true.

(1.21) Theorem (Banach - Stone). Let X and Y be compact Hausdorff spaces. If $g: X \longrightarrow Y$ is given, define $g^*: C(Y) \longrightarrow C(X)$ by $g^*(h)(x) = h(g(x))$. Then g^* is an isomorphism onto iff g is a homeomorphism onto.

The chief result of this section, 1.21', generalizes this theorem.

Let Z be a space and recall that $S(Z,\underline{R}) = \{P \subset Z \times \underline{R} \mid Z = P\underline{R}\}$. Let us define addition and multiplication in $S(Z,\underline{R})$ as follows: if A and B are subsets of $\underline{R}$, let $A + B = \{a + b \mid a \varepsilon A \text{ and } b \varepsilon B\}$ and $AB = \{ab \mid a \varepsilon A \text{ and } b \varepsilon B\}$, and then if P and Q are any two elements of $S(Z,\underline{R})$, define $P + Q = \cup \{\{z\} \times (zP + zQ) \mid z \varepsilon Z\}$ and $PQ = \cup \{\{z\} \times ((zP)(zQ)) \mid z \varepsilon Z\}$.

If X and Y are spaces and $f: X \longrightarrow Y$, let us define f^* as follows: for each $Q \varepsilon S(Y,\underline{R})$, let $f^*(Q) = \cup \{\{x\} \times ((f(x))Q \mid x \varepsilon X\}$. Then $f^*(Q) \subset X \times \underline{R}$ by definition; $Q \varepsilon S(Y,\underline{R})$ implies $yQ \neq \Box$ for each y in Y, hence $x(f^*(Q)) = (f(x))Q \neq \Box$ for each x in X. Therefore $f^*(Q) \varepsilon S(X,\underline{R})$: in other words, f^* is a function from $S(Y,\underline{R})$ into $S(X,\underline{R})$.

(1.22) <u>Lemma</u>. If X and Y are spaces, $f: X \longrightarrow Y$ is a function and $f^*: S(Y,\underline{R}) \longrightarrow S(X,\underline{R})$ is defined as above, then

(i) f^* is a homomorphism;

(ii) if f is continuous, $f^*(C(Y,\underline{R})) \subset C(X,\underline{R})$; that is, if $P \subset Y \times \underline{R}$ is such that $Y = P\underline{R}$, P is u.s.c. and l.s.c. on Y and point compact, then $f^*(P)$ is such that $X = (f^*(P))\underline{R}$, $f^*(P)$ is u.s.c. and l.s.c. on X and point compact.

Proof: (i) Let P and Q be any elements of $S(Y,\underline{R})$. Then $f^*(P + Q) = f^*(P) + f^*(Q)$: for let $x \in X$ and note $x(f^*(P + Q)) = (f(x))(P + Q) = (f(x))P + (f(x))Q = x(f^*(P)) + x(f^*(Q)) = x(f^*(P + Q))$, all by definition either of f^* or of addition in $S(Y,\underline{R})$. Similarly, $f^*(PQ) = (f^*(P))(f^*(Q))$.

(ii) Suppose f is continuous and let $P \in C(Y,\underline{R})$. Define $p: Y \longrightarrow 2^{\underline{R}}$ by $p(y) = yP$ for each y in Y; since P is u.s.c. and l.s.c., by 1.8 p is continuous. The graph of pf in $X \times \underline{R}$ is exactly $f^*(P)$, since $pf(x) = (f(x))P = x(f^*(P))$ for each x in X. Since pf is continuous, by 1.8 again, $f^*(P)$ is u.s.c. and l.s.c. on X. Finally, $x(f^*(P)) = (f(x))P$, which is compact since $f(x) \in Y$ and since $P \in C(Y,\underline{R})$ implies yP compact for every y in Y.

(1.21') <u>Theorem</u>. Let X and Y be compact, Hausdorff and nonnull spaces. Let $\underline{R}$ be the real numbers, let $C(X,\underline{R}) = \{P \subset X \times \underline{R} \mid X = P\underline{R},$ P is u.s.c. and l.s.c. on X and point compact$\}$, and let $C(Y,\underline{R}) = \{Q \subset Y \times \underline{R} \mid Y = Q\underline{R}, Q$ is u.s.c. and l.s.c. on Y and point compact$\}$.

Define f^*: $C(Y,\underline{R}) \longrightarrow C(X,R)$ by $x(f^*(Q)) = (f(x))Q$, for Q in $C(Y,\underline{R})$, for a given f: $X \longrightarrow Y$. Then

(i) f^* is 1-1 iff f is onto, and

(ii) f^* is onto iff f is 1-1.

Therefore, f is a homeomorphism iff f^* is an isomorphism.

Proof: f^* is a homomorphism by 1.22(i).

(i) Suppose f is onto and let $P \neq Q$ in $C(Y,\underline{R})$. Then there is some y in Y such that $yP \neq yQ$, and since f is onto, there is some x in $f^{-1}(y)$. Since $f(x) = y$, $x(f^*(P)) = yP$ and $x(f^*(Q)) = yQ$, and therefore $f^*(P) \neq f^*(Q)$.

Conversely, suppose f^* is 1-1. Since f is continuous and X is compact, $f(X)$ is compact; since Y is Hausdorff, $f(X)$ is closed in Y; and since Y is compact and Hausdorff, Y is normal and points are closed. Therefore, if there is any point $y \notin f(X)$, by Urysohn's Lemma, there is a continuous function q: $Y \longrightarrow \underline{R}$ such that $q(y) = 0$ and $q(f(X)) = 1$. Let Q be the graph of q. Let p: $Y \longrightarrow \underline{R}$ be such that $p(Y) = 1$, and let P be the graph of p. It is easily seen that the graph of any continuous single-valued function from Y to $\underline{R}$ is in $C(Y,\underline{R})$. Since $yP = 1$ and $yQ = 0$, $P \neq Q$; however, $f^*(P) = f^*(Q)$: for let $x \in X$; since $f(x) \in Y$, $(f(x))P = 1$, and since $f(x) \in f(X)$, $(f(x))Q = 1$. This contradicts the hypothesis that f^* is 1-1, so we can conclude that $Y \setminus f(X) = \Box$: that is, f is onto.

(ii) Suppose f^* is onto and let $x \neq x'$ in X. Again Urysohn's Lemma can be applied to find a continuous function t: $X \longrightarrow \underline{R}$ such that

$t(x) = 0$ and $t(x') = 1$. Let T be the graph of t and note that $T \in C(X,\underline{R})$. By hypothesis, there is some $Q \in C(Y,\underline{R})$ such that $f^*(Q) = T$. In particular, $x(f^*(Q)) = xT$ and $(x')(f^*(Q)) = (x')T$, which implies $(f(x))Q = xT = 0$ and $(f(x'))Q = (x')T = 1$. Therefore $f(x) \neq f(x')$, and since x and x' were arbitrary distinct points of X, we have proved that f is 1-1.

Conversely, suppose f is 1-1. Note f^{-1}: $f(X) \longrightarrow X$ is a continuous function: for f 1-1 implies f^{-1} single-valued; and X compact and f continuous imply $f = (f^{-1})^{-1}$ closed, so f^{-1} is continuous. To prove the assertion, let $Q \in C(X,\underline{R})$; we will construct $P \in C(Y,\underline{R})$ such that $f^*(P) = Q$. By 1.17, XQ is compact, so there is a closed interval $I \subset \underline{R}$ such that $XQ \subset I$. Define q: $X \longrightarrow \overline{2}^{I}$ by $q(x) = xQ$ for each x in X, and note that 1.8 implies q is continuous. By Theorem 3 and Corollary 3.1 of [16], the space of closed subsets of a closed interval of $\underline{R}$ is a CAR*: i.e., since Y is normal, $f(X)$ closed in Y and qf^{-1}: $f(X) \longrightarrow \overline{2}^{I}$ is continuous, there is a continuous extension p of qf^{-1} to all of Y. $p(Y) \subset \overline{2}^{I}$ implies $p(y)$ compact in $\underline{R}$ for each y in Y. So if P is the graph of p in $Y \times \underline{R}$, we have $Y = P\underline{R}$ and yP compact for each y in Y. Also, p is continuous, so by 1.8 again, P is u.s.c. and l.s.c. on X. Therefore $P \in C(Y,\underline{R})$. Finally, $f^*(P) = Q$: for let $x \in X$; then $x(f^*(P)) = (f(x))P = pf(x)$, and $p = qf^{-1}$ on $f(X)$ so $pf(x) = qf^{-1}f(x) = q(x) = xQ$. That is, $f^*(P) = Q$, and this completes the proof that $f^*(C(Y,\underline{R})) = C(X,\underline{R})$.

PART II

SECTION 1. Background and definitions.

Let X and Y be spaces. A function $f\colon X \longrightarrow Y$ is said to be monotone iff $f^{-1}(y)$ is connected for each y in Y. This is a very restrictive property, and continuous monotone functions preserve many topological properties. For example, if $f\colon X \longrightarrow Y$ is a continuous and monotone function onto a nondegenerate Hausdorff space, it is known and easily proved that

(1) X an arc implies Y an arc;

(2) X a pseudocircle implies Y a pseudocircle;

(3) X a tree implies Y a tree; and

(4) X unicoherent implies Y unicoherent.

We will call a compact connected Hausdorff space a <u>continuum</u>. A <u>cutpoint</u> of a space X is a point x in X such that $X \setminus x$ is not connected. By an <u>arc</u> we mean a continuum with exactly two noncutpoints. Notice that an arc need not be metric under this definition; an example of a nonmetric arc is the "long line" (see [7], for example). A <u>pseudocircle</u> is defined to be a nondegenerate continuum which is disconnected by any two of its points. A <u>tree</u> is a continuum in which each two points are separated by a third point. A space is <u>unicoherent</u> iff it is a continuum and any two subcontinua whose union is the whole space have a connected intersection.

See [22] for proofs of (1) - (4) in the case where all spaces are assumed metric. In the following, we will prove generalizations of (1) - (3). So far we have not found a relation to preserve unicoherence.

Let $R \subset X \times Y$ and define R to be monotone iff Ry is connected for each y in Y. If R is the graph of a single-valued function $f: X \longrightarrow Y$, then $Ry = f^{-1}(y)$, so R is a monotone relation iff f is a monotone function. Thus monotonicity for relations is a generalization of monotonicity for functions. Notice that R is the graph of a single-valued function iff $X = RY$ and $Ry \cap Ry' = \square$ for each pair of distinct points, y and y', in Y.

It is easy to exhibit an u.s.c., l.s.c. and monotone relation which does not preserve unicoherence or the properties of being an arc, tree or pseudocircle. The problem seems to be that requiring $R \subset X \times Y$ to be monotone and u.s.c. and l.s.c. on X does not in any way distinguish Ry from Ry' for any y and y' in Y. For example, let X be any connected space, let S be a pseudocircle and define $R \subset X \times Y$ by $xR = S$ for each x in X. It is easily seen that R is monotone and u.s.c. and l.s.c. on X, but X could be unicoherent, an arc or a tree and $XR = S$ is none of these things. Notice that $Rs = X$ for each s in S. A similar relation can be defined in $S \times I$, where I is an arc, so show that the property of being a pseudocircle is not preserved by a relation which is monotone and u.s.c. and l.s.c. on S.

Now suppose that I is an arc, Y is a nondegenerate Hausdorff space, and $R \subset I \times Y$ is such that $I = RY$. Conditions are known for

R which imply that IR is a continuum (see [13] or [15]), and it is well known that a nondegenerate continuum has at least two noncutpoints. Thus some additional conditions are needed which will imply that IR has at most two noncutpoints. We discovered that it suffices to require R to be monotone and <u>noninclusive</u>: that is, for each $y \neq y'$ in Y, $Ry \not\subset Ry'$. Notice that the graph of a single-valued, onto function is noninclusive.

Several lemmas are given before we prove 2.6 and 2.6', the arc theorem in two forms. Theorems 2.8, 2.15 and 2.15', which exhibit relations that preserve pseudocircles and trees, are essentially applications of 2.6 and 2.6', using the facts that a pseudocircle is a certain union of arcs, and certain subcontinua of a tree are arcs.

SECTION 2. Arcs and metric arcs.

If $R \subset X \times Y$, let $R^{(-1)} = \{(y,x) \mid (x,y) \in R\}$. Then $R^{(-1)}$ is a relation in $Y \times X$, and $R^{(-1)}x = xR$ and $yR^{(-1)} = Ry$.

Notice that saying $R^{(-1)}$ is monotone is just saying that xR is connected for each x in X. Strother calls such a relation <u>point connected</u>.

(2.1) <u>Lemma</u>. Let I and Y be spaces, let Y be nondegenerate and let $R \subset I \times Y$.

(i) If R is noninclusive, then $IR = Y$.

(ii) R is noninclusive iff for each y in Y, $(I \setminus Ry)R = Y \setminus y$.

Proof: (i) Suppose R is noninclusive and $y \in Y \setminus IR$. Then $Ry = \square$; since Y is nondegenerate, there is some $y' \in Y \setminus y$, but then $Ry' \supset \square = Ry$, which contradicts the noninclusivity of R. Therefore $Y = IR$.

(ii) For any relation $R \subset I \times Y$, $(I \setminus Ry)R \subset Y \setminus y$; so suppose R is noninclusive and let $y' \in Y \setminus y$. Since $Ry' \not\subset Ry$, there is some x in $Ry' \cap (I \setminus Ry)$; then $y' \in xR \subset (I \setminus Ry)R$.

Conversely, suppose for each y in Y, $(I \setminus Ry)R = Y \setminus y$, and let $y \neq y'$ in Y. Then $y' \in Y \setminus y$; if $Ry' \subset Ry$, then $y' \notin (I \setminus Ry)R$, which is false. Therefore $Ry' \not\subset Ry$.

The equivalence for noninclusivity given in 2.1 (ii) is the only one found so far.

Recall that $R \subset I \times Y$ is point compact iff for each x in I, xR is compact.

(2.2) Lemma. Let I and Y be Hausdorff spaces and let $R \subset I \times Y$ be such that $I = RY$, R is u.s.c. on I and point compact. Then

(i) if B^* is compact in I, $(BR)^* \subset (B^*)R$, and

(ii) for each y in Y, Ry is closed.

Proof: (i) Certainly $(BR)^* \subset ((B^*)R)^*$, and by 1.17, $(B^*)R$ is compact. Y is Hausdorff, so $(B^*)R$ is closed, and therefore $(BR)^* \subset ((B^*)R)^* = (B^*)R$.

(ii) By Theorem 2.9 (d) of [13], R is closed; $R^{(-1)}$ is homeomorphic to R, hence $R^{(-1)}$ is closed. Then by Theorem 2.11 of [13], $yR^{(-1)} = Ry$ is closed for each y in Y.

The following facts are well known; it is convenient for us to use [7] for reference. Let I be a connected space with exactly two noncutpoints, a and b, and let I have topology $\mathcal{J}$. The cutpoint ordering for I is defined as follows: for any p and q in I, define $p \leq q$ iff $p = a$, $p = q$ or p separates a and q. By Theorem 2-21 of [7], this is a simple, i.e., linear, order on I. If we define $p < q$ to mean $p \leq q$ and $p \neq q$, $[p,q) = \{x \in I \mid p \leq x < q\}$ and $(p,q] = \{x \in I \mid p < x \leq q\}$, then $\{[a,p)$ and $(q,b] \mid p \cup q \subset I \setminus (a \cup b)\}$ is a subbasis for the cutpoint order topology for I, which we will call $\mathcal{O}$. By Theorem 2-24 of [7], $\mathcal{O} \subset \mathcal{J}$. When I is also compact and Hausdorff, i.e., when I is an arc, $\mathcal{O} = \mathcal{J}$ by Theorem 2-25 of [7].

Henceforth in this paper, whenever we assume I is an arc with noncutpoints a and b, we will suppose that the cutpoint order and topology have been defined on I as above, and we will use the fact that the topology of I is the order topology. In addition to the notation defined above, we will write (p,q) to denote $\{x \in I \mid p < x < q\}$ and $[p,q]$ to denote $\{x \in I \mid p \leq x \leq q\}$.

(2.3) Theorem. Let I be an arc with noncutpoints a and b, let Y be a nondegenerate Hausdorff space, and let $R \subset I \times Y$ be such that $I = RY$, R is u.s.c. on I, point compact, monotone and noninclusive. Then

(i) $IR = y$,

(ii) Y is compact,

(iii) $aR = a_y \neq b_y = bR$, where a_y and b_y are points of Y, and

(iv) each y in $Y \setminus (a_y \cup b_y)$ is a cutpoint of Y.

Proof: (i) Since R is noninclusive, by 2.1 (i), $IR = Y$.

(ii) Since $I = RY$, I is compact, and R is u.s.c. on I and point compact, by 1.17, IR is compact: that is, Y is compact.

(iii) To prove aR is a single point, suppose $aR \supset y_1 \cup y_2$. Since R is monotone and by 2.2 (ii), Ry_1 and Ry_2 are closed and connected; so there are points x_1 and x_2 in I such that $Ry_1 = [a,x_1]$ and $Ry_2 = [a,x_2]$. We may suppose $x_1 \leq x_2$, which implies $Ry_1 \subset Ry_2$; since R is noninclusive, this implies $y_1 = y_2$. Since $I = RY$, $aR \neq \square$, which completes the proof that aR is exactly one point. Similarly bR is a point. Let $aR = a_y$ and $bR = b_y$.

Suppose $a_y = b_y$: that is, $aR = bR$. Then $a \cup b \subset Ra_y = Rb_y$; Ra_y is connected by hypothesis, and I is irreducibly connected between a and b, so $Ra_y = I$. But by hypothesis there is at least one y in $Y \setminus a_y$ and of course $Ry \subset I$. This contradicts the noninclusivity of R, and hence $a_y \neq b_y$.

(iv) Let $y \varepsilon Y \setminus (a_y \cup b_y) = Y \setminus (aR \cup bR)$; then $Ry \subset (a,b)$. By 2.2 (ii) and since R is monotone, $Ry = [p,q]$. Let $P = [a,p)$, $Q = (q,b]$, and note $P \neq \square \neq Q$. By 2.1 (ii), $(I \setminus Ry)R = Y \setminus y$: that is, $PR \cup QR = Y \setminus y$. Finally, PR and QR are nonnull separated sets. For suppose $y' \varepsilon (PR)^* \cap QR$; by 2.2 (i), $y' \varepsilon (P^*)R \cap QR$, which says $Ry' \cap P^* \neq \square$ and $Ry' \cap Q \neq \square$. Then the structure of I and the fact that Ry' is connected imply $Ry' \supset [p,q]$, and hence, since R is non-inclusive, $y' = y$. But $Ry \cap Q = \square$ by definition of Q, so $y' \neq y$. Therefore no such y' exists. Similarly, $PR \cap (QR)^* = \square$. PR and QR are each nonnull since P and Q are nonnull and since $I = RY$. Therefore $Y \setminus y$ is the union of two nonnull separated sets, which says y is a cutpoint of Y.

(2.4) Lemma. Let I and Y be arcs such that the noncutpoints of I are a and b and those of Y are a_y and b_y. For p and q in I, define $p \leq q$ iff $p = a$, $p = q$ or p separates a and q; for y and z in Y, define $y \leq z$ iff $y = a_y$, $y = z$ or y separates a_y and z. Let $R \subset I \times Y$ and suppose:

(i) R is noninclusive;

(ii) if A is connected in I, AR is connected;

(iii) $aR = a_y$ and $bR = b_y$;

(iv) $y_1 < y_2$ in Y; and

(v) $Ry_1 = [p_1,q_1]$ and $Ry_2 = [p_2,q_2]$.

Then if $x \varepsilon [a,p_i)$, $xR \subset [a_y,y_i)$, and if $x \varepsilon (q_i,b]$, $xR \subset (y_i,b_y]$, for $i = 1, 2$; also $p_1 < p_2$ and $q_1 < q_2$.

Proof: Let $P = [a,p_1)$. Since $P \subset I \setminus Ry_1$, $PR \subset Y \setminus y_1 = [a_y,y_1) \cup (y_1 b_y]$. By (ii), PR is connected; also $a \varepsilon P$, so $PR \subset [a_y,y_1)$. Similarly, if $x \varepsilon [a,p_2)$, $xR \subset [a_y,y_2)$ and if $x \varepsilon (q_i,b]$, $xR \subset (y_i,b_y]$ for $i = 1, 2$.

Remark: since (iv) implies $y_1 \neq y_2$, (i) and (v) imply that either $p_1 < p_2$ and $q_1 < q_2$ or $p_2 < p_1$ and $q_2 < q_1$. Suppose the latter is true; then $a \leq p_2 < p_1$. By (iv), $y_1 < y_2$, and by the previous paragraph, $PR \subset [a_y,y_1)$; hence $y_2 \not\varepsilon PR$, which implies $P \cap Ry_2 = \square$. But $p_2 \varepsilon Ry_2$ and by supposition, $p_2 \varepsilon [a,p_1) = P$. This is a contradiction, and it follows that $p_1 < p_2$ and $q_1 < q_2$.

The following lemma is due to Strother; he states it for R either u.s.c. or l.s.c. on I, as Theorem 3.9 in [13], but we need only the proof for R u.s.c. on I.

(2.5) Lemma (Strother). Let $R \subset I \times Y$ and $I = RY$; let R be u.s.c. on I and $R^{(-1)}$ be monotone. Then if A is connected in I, AR is connected in Y.

Proof: Let A be connected in I and suppose $AR = P \cup Q$, where P and Q are nonnull separated sets. Then for each a in A, aR is connected so either $aR \subset P$ or $aR \subset Q$. Also, neither A_P nor A_Q is null, where $A_P = \{a \varepsilon A \mid aR \subset P\}$ and $A_Q = \{a \varepsilon A \mid aR \subset Q\}$; and $A_P \cap A_Q = \square$. Since R is u.s.c., $\{a \varepsilon A \mid aR \subset Y \setminus Q^*\}$ is open in A and $\{a \varepsilon A \mid aR \subset Y \setminus P^*\}$ is open in A; but these are just A_P and A_Q, respectively, which contradicts the fact that A is connected. Therefore AR must be connected.

The next two theorems describe monotone relations which preserve the arc. Actually, the two sets of hypotheses are equivalent, which is proved by 2.6 (iv) and 2.6' (iv).

(2.6) Theorem. Let I be an arc, let Y be a nondegenerate Hausdorff space, and let $R \subset I \times Y$ be such that $I = RY$, R is point compact, u.s.c. on I, monotone, noninclusive, and $R^{(-1)}$ is monotone. Then

(i) $IR = y$,

(ii) $aR = a_y \neq b_y = bR$, where a_y and b_y are points of Y,

(iii) Y is an arc whose noncutpoints are a_y and b_y, and

(iv) R is l.s.c. on I.

Proof: (i) and (ii) follow from (i) and (iii) of 2.3.

(iii) By 2.3 (ii), Y is compact; by 2.5, Y is connected; and by hypothesis Y is Hausdorff; so Y is a continuum. By 2.3 (iii),

Y is nondegenerate. It is well known that a nondegenerate continuum has at least two noncutpoints. By 2.3 (iv), the only possible noncutpoints of Y are a_y and b_y; hence we can conclude that Y is an arc with noncutpoints a_y and b_y.

(iv) To prove R l.s.c., it will suffice to let U be any subbase element of Y and prove RU open. So let y_0 be any element of $Y \setminus (a_y \cup b_y)$ and let $U = (y_0, b_y]$. Let $x \varepsilon RU$ and find y in U such that $x \varepsilon Ry$. Let $Ry = [p,q]$. Since $y_0 < y$ and Y is connected, there is some $y_1 \varepsilon Y$ such that $y_0 < y_1 < y$; let $Ry_1 = [p_1, q_1]$. Then by 2.4, $p_1 < p$ and $q_1 < q$, so $x \varepsilon [p,q] \subset (p_1, b_y]$, which is open in I. Finally, $(p_1, b_y] \subset RU$; for let $x' \varepsilon (p_1, b_y]$. If $x' \varepsilon (p_1, q_1]$, then $y_1 \varepsilon (x')R \cap U$; and if $x' \varepsilon (q_1, b_y]$, by 2.4, $(x')R \subset (y_1, b_y] \subset U$.

If we let $V = [a_y, y_0)$, it is clear that 2.4 and a dual argument will imply RV open. Therefore, R is l.s.c. on I.

(2.6') Theorem. Let I be an arc, let Y be a nondegenerate Hausdorff space, and let $R \subset I \times Y$ be such that $I = RY$, R is point compact, u.s.c. on I, monotone, noninclusive, and R is l.s.c. on I. Then

(i) $IR = Y$,

(ii) $aR = a_y \neq b_y = bR$, where a_y and b_y are points of Y,

(iii) Y is an arc whose noncutpoints are a_y and b_y, and

(iv) $R^{(-1)}$ is monotone.

Proof: (i) and (ii) follow from (i) and (iii) of 2.3.

(iii) By 2.3(ii), Y is compact, and by hypothesis, Y is

Hausdorff. Define $f: I \longrightarrow 2^Y$ by $f(x) = xR$ for each x in I; since R is u.s.c. and l.s.c. on I, by 1.8, f is continuous. I is connected, hence $f(I)$ is connected in 2^Y, and $f(a) = a_y$, a connected subset of Y. Then Proposition 2.8 of [11] applies, and we can conclude that $\cup \{f(x) \mid x \in I\} = IR = Y$ is connected. Therefore Y is a continuum, and as in 2.6, we can conclude that Y is an arc with noncutpoints a_y and b_y.

(iv) Now that we know that Y is an arc with noncutpoints a_y and b_y, we can define $y \leq z$ in Y iff $y = a_y$, $y = z$ or y separates a_y and z, and we know the order topology is the topology of Y. We want to prove xR connected for each x in I. If $x \in I$ and xR is a point, we are done. Otherwise, suppose $x \in I$ and $xR \supset y_1 \cup y_2$, where $y_1 < y_2$. It will suffice to prove that $xR \supset (y_1, y_2)$, so let $y \in (y_1, y_2)$. Since R is monotone and by 2.2 (ii), we may suppose $Ry_1 = [p_1, q_1]$, $Ry = [p, q]$ and $Ry_2 = [p_2, q_2]$. By 2.4, since $y_1 < y < y_2$, we have $p_1 < p < p_2$ and $q_1 < q < q_2$. We know that $x \in Ry_1 \cap Ry_2$, hence $p_2 \leq x \leq q_1$. Then $x \in (p,q) \subset Ry$, which implies $y \in xR$.

Now suppose that I is an arc, Y is a nondegenerate Hausdorff space, and $f: I \longrightarrow Y$ is a continuous, monotone, single-valued function onto Y. As mentioned on page 30, it is known that this implies that Y is an arc, and that known theorem is a special case of 2.6. (It is easy to see that the graph of f is a relation which satisfies all the hypotheses of 2.6.)

If $R \subset X \times Y$, Strother calls $f\colon X \longrightarrow Y$ a <u>trace</u> for R iff f is continuous and $f(x) \in xR$ for each x in X. In the next theorem, we define a trace for the relations of 2.6 and 2.6', and use it to prove that those relations preserve metricity as well as the arc. In other words, either of those relations takes a metric arc onto a metric arc. It is not difficult to see that a metric arc is just a homeomorph of the unit interval. (A proof can be found under 2-27 of [7].)

(2.7) <u>Theorem</u>. Let I be a metric arc, Y a nondegenerate Hausdorff space, and let $R \subset I \times Y$ be such that $I = RY$, R is point compact, u.s.c. on I, monotone, noninclusive, and either $R^{(-1)}$ is monotone or R is l.s.c. on I. Then Y is a metric arc.

Proof: If $R^{(-1)}$ is monotone, by 2.6, Y is an arc and R is l.s.c. on I. If R is l.s.c. on I, by 2.6', Y is an arc. Therefore we only need to prove that Y is metric.

Let a and b be the noncutpoints of I; by 2.6 (ii), $aR = a_y$ and $bR = b_y$ and by 2.6 (iii), a_y and b_y are the noncutpoints of Y. Let us define $y \leq z$ in Y iff $y = a_y$, $y = z$ or y separates a_y and z. Then the topology of Y is the order topology; the order is linear; and every closed set has a first element, since g.l.b.(A) exists for each $A \subset Y$ (see, for example, Theorem 2-26 of [7]).

Theorem 1.9 of [11] says that in such case there is a continuous function $g\colon \overline{2}^{Y} \longrightarrow Y$ such that $g(A) \in A$ for each A in S. Since R is point compact and Y is Hausdorff, R is point closed, so we can define $f\colon I \longrightarrow \overline{2}^{Y}$ by $f(x) = xR$ for each x in I.

Since R is u.s.c. and l.s.c. on I, by 1.8, f is continuous. Therefore $gf\colon I \longrightarrow Y$ is continuous, and hence $gf(I)$ is connected. By definition of g and f, $gf(a) = a_y$ and $gf(b) = b_y$, so $a_y \cup b_y \subset gf(I)$; Y is irreducibly connected between a_y and b_y, hence $gf(I) = Y$. Finally, since I is a compact metric space and gf is continuous, $gf(I)$ is metric (see Theorem 3-23 in [7] for example). This completes the proof.

SECTION 3. Pseudocircles and simple closed curves.

Recall that S is a pseudocircle iff S is a nondegenerate continuum which is separated by any two of its points. When a pseudocircle is metric, it is a homeomorph of the unit circle; that is, it is a simple closed curve. Theorem 2-28 of [7] gives a proof of this.

(2.8) Theorem. Let S be a pseudocircle, Y a Hausdorff space and $a \neq b$ in S. Let $R \subset S \times Y$ be such that $S = RY$, R is point compact, u.s.c. on S, monotone, noninclusive, and either $R^{(-1)}$ is monotone or R is l.s.c. on S. Let $aR = a_y \neq b_y = bR$, where a_y and b_y are points of Y. Then Y is a pseudocircle, and if S is metric, Y is metric.

Proof: Since S is compact, by lemma 11.19 of [23], S is the union of two arcs whose intersection is their noncutpoints, a and b. Let us call these arcs I and J.

By 2.1 (i), $SR = Y$, so $IR \cup JR = Y$. Let $P = R \cap (I \times S)$ and $Q = R \cap (J \times S)$, and let us consider P as a relation in $I \times IR$ and Q as a relation in $J \times JR$.

Notice that for any x in I, $xP = xR$, and for any y in IR, $Py = Ry \cap I$. Then it is obvious that P is u.s.c. on I, point compact, and if $R^{(-1)}$ is monotone, $P^{(-1)}$ is monotone. To see that P is monotone, let $y \in IR$ and suppose Py is not connected. Since Ry is connected and $Py = Ry \cap I$, the structure of S implies that both a and b lie in RY.

But then $y \in aR \cap bR$, which is not true. Therefore Py must be connected. Finally, P is noninclusive; for let $y \neq z$ in IR and suppose $Py \subset Pz$. Then $Ry \not\subset I$, else $Ry = Py \subset Pz \subset Rz$, contradicting the noninclusivity of R. Ry is connected and intersects I, so one of a or b must lie in Ry, hence in Py. But $a \in Py \subset Pz$ and $aP = aR$ imply $aR \supset y \cup z$, which is false; similarly, $b \notin Py$. This involves a contradiction, so we can conclude that $Py \not\subset Pz$, for any $y \neq z$ in IR.

Dually, Q is point compact, u.s.c. on J, monotone, noninclusive, if $R^{(-1)}$ is monotone, $Q^{(-1)}$ is monotone, and if R is l.s.c. on S, Q is l.s.c. on J.

Then if $R^{(-1)}$ is monotone, IP and JQ are arcs by 2.6; if R is l.s.c. on S, IP and JQ are arcs by 2.6'; and in either case, the non-cutpoints of IP and JQ are a_y and b_y. Further, $IP \cap JQ = a_y \cup b_y$: for if $y \in IP \cap JQ$, Ry intersects both I and J; Ry is connected since R is monotone; so the structure of S implies that either $a \in Ry$ or $b \in Ry$. If $a \in Ry$ then $y = a_y$ and if $b \in Ry$, $y = b_y$, by hypothesis.

This completes the proof that Y is a pseudocircle, since $Y = IP \cup JQ$ implies Y is a continuum, and surely any two points of Y disconnect Y.

Finally, suppose S is metric; then I and J are metric so by 2.7, IP and IQ are metric arcs. It is clear that Y is then homeomorphic to the unit circle.

It is quite possible that the hypotheses of 2.8 can be weakened so that the relation need not be single-valued at any point of S. Consider the following example.

Let S be the unit circle in the complex plane and define $R \subset S \times S$ as follows: $zR = \{z' \mid \text{amp } z \leq \text{amp } z' \leq \text{amp } z + \pi\}$. It is obvious that R is point compact, u.s.c. and l.s.c. on I, and $R^{(-1)}$ is monotone. Also, R is noninclusive and monotone, since $Rz' = \{z \mid \text{amp } z' - \pi \leq \text{amp } z \leq \text{amp } z'\}$ for any z' in S. Thus R satisfies all the hypotheses of 2.8 except that zR is not a point for any z.

SECTION 4. Trees and dendrites.

Recall that a tree is a continuum in which every two points are separated by a third point.

In [20], L. E. Ward, Jr. has given a characterization of trees in terms of partial order. We will use this theorem extensively, so it is quoted below. First several definitions are needed.

A partial order is a binary relation which is reflexive, transitive and anti-symmetric. If $(X,\leq)$ is a partially ordered set, then for any x in X we define $\underline{L(x)} = \{y \mid y \text{ in } X \text{ and } y \leq x\}$, and $\underline{M(x)} = \{y \mid y \text{ in } X \text{ and } x \leq y\}$. $\leq$ is said to be semi-continuous iff $L(x)$ and $M(x)$ are closed for each x in X. $\leq$ is said to be order dense iff for each two distinct points of X, x and y, there is some z in X such that $x < z < y$. A is a chain in $(X,\leq)$ iff for each a and a' in A either $a \leq a'$ or $a' \leq a$.

(2.9) Theorem (Ward). Let X be a compact Hausdorff space. X is a tree iff X has a partial order $\leq$ such that

(i) $\leq$ is semi-continuous;

(ii) $\leq$ is order dense;

(iii) for each x and y in X, $L(x) \cap L(y)$ is a nonnull chain;

(iv) for each x in X, $M(x) \setminus x$ is open.

To prove necessity, Ward lets X be a tree and defines $\leq$ thus: choose an arbitrary e in X; then for any x and y in X, define $x \leq y$

iff $x = e$, $x = y$, or x separates e and y. Ward shows that $\leq$ is a partial order on T which satisfied (i) - (iv), whatever point e may be.

If T is a space and $a \neq b$ in T, we will let $C(a,b)$ denote $\{t \in T \mid t \text{ separates } a \text{ and } b \text{ in } T\} \cup a \cup b$. The following lemma is known; it is an easy result of two of Ward's theorems.

(2.10) Lemma. If T is a tree and $a \neq b$ in T, $C(a,b)$ is an arc with noncutpoints a and b.

Proof: Define $x \leq y$ in T iff $x = a$, $x = y$ or x separates a and y. Then $L(b) = C(a,b)$; $L(b)$ is closed by 2.9 (i), and hence is compact; the order on $L(b)$ is semicontinuous by 2.9 (i); and $L(b)$ is a chain by 2.9 (iii). $L(b)$ is order dense since, if $x < y$ in $L(b)$, there is some z in T such that $x < z < y$ by 2.9 (ii), and $z < y$ implies $z \in L(b)$. Then by Theorem 5 of [19], $L(b)$ is connected. $L(b)$ is Hausdorff since T is, so $L(b)$ is a continuum: i.e., $C(a,b)$ is a continuum. By definition, $C(a,b)$ has at most two noncutpoints; it must have at least two, hence $C(a,b)$ is an arc and the noncutpoints of $C(a,b)$ are exactly a and b.

The following lemma is known. See, for example, p. 74 of [24].

(2.11) Lemma. If $a \neq b$ in a tree T, if X is a connected subset of T and if $a \cup b \subset X$, then

(i) $C(a,b) \subset X$, and

(ii) x separates a and b in X iff $x \in C(a,b)$.

Proof: (i) If $x \notin X$, then $T \setminus x \supset X$, which is a connected set containing a and b. Therefore $x \notin C(a,b)$.

(ii) Suppose $X \setminus x = P \cup Q$, separated sets such that $a \in P$ and $b \in Q$. By (i), $C(a,b) \subset X$, and by 2.10, $C(a,b)$ is connected. Thus if $x \notin C(a,b)$, either $C(a,b) \subset P$ or $C(a,b) \subset Q$; but neither of these is true since $a \in C(a,b) \cap P$ and $b \in C(a,b) \cap Q$. Therefore $x \in C(a,b)$. The converse is obvious.

A <u>component</u> of a space X is a maximal connected subset of X. A <u>branch point</u> of a tree T is a point t such that $T \setminus t$ has three or more components.

The next lemma is known; in the case of metric spaces, it can be deduced from 1.1 (iv) of [22].

(2.12) <u>Lemma</u>. If T is a tree, B is the set of all branch points of T, and J is a nondegenerate connected set in $T \setminus B$, then J^* is an arc.

Proof: J^* is a nondegenerate continuum, so it has at least two noncutpoints. To prove there are exactly two, suppose the contrary and let a, b and c be distinct noncutpoints of J^*.

Define $x \leq y$ in T iff $x = a$, $x = y$ or x separates a and y. Let $X = L(b) \cup L(c)$. Since $J^* \setminus c$ is connected and $a \cup b \subset J^* \setminus c$,

$c \notin C(a,b) = L(b)$. Therefore $c \in X \setminus L(b)$. Similarly $b \in X \setminus L(c)$. By 2.9 (iii), $L(b) \cap L(c)$ is a chain; the order is anti-symmetric, so $L(b) \cap L(c)$ contains at most one maximum element. By 2.9 (i) and the compactness of T, $L(b) \cap L(c)$ is a compact chain with semicontinuous order, so it contains a maximum element by a remark by Wallace in [18]. Let $x = \max (L(b) \cap L(c))$, and notice that $X \setminus M(x) \subset L(b) \cap L(c)$. Also $a \in X \setminus M(x)$: for $a \leq x$, and if $a = x$, then a separates b and c; but this is false since $J^* \setminus a$ is a connected set containing b and c.

Finally, $X \setminus x = (X \setminus M(x)) \cup (X \setminus L(b)) \cup (X \setminus L(c))$, and these sets are easily seen to be disjoint. Also, each is open in X since points are closed and the order is semicontinuous. Therefore x separates a, b and c pairwise in X; X is connected so by 2.11 (ii), x separates them pairwise in T. Thus x is a branch point of T. But $x \in J$: for $J \cup a \cup b$ is connected, so by 2.11 (i), $C(a,b) \subset J \cup a \cup b$; $x \in C(a,b)$ and $x \notin a \cup b$, hence $x \in J$. Since J was chosen in the complement of the set of branch points of T, we have reached a contradiction. It follows that J^* contains exactly two noncutpoints and is therefore an arc.

The following lemma is known; actually, both parts are special cases of more general theorems (see Theorem 66 of [12]), but we need only these weaker statements which are easily proved using the fact that $C(a,b)$ is connected for any points a and b in a tree.

(2.13) Lemma. Let A and B be nonnull subcontinua of a tree T.

(i) If $A \cap B = \square$, there is some t in T such that t separates A and B.

(ii) If $A \cap B \neq \square$ and if $A \setminus B$ and $B \setminus A$ are nonnull connected sets, then $A \cap B$ separates $A \setminus B$ and $B \setminus A$.

Proof: (i) Let $a \in A$ and $b \in B$. By 2.10, $C(a,b)$ is connected, and it intersects both A and B, so there is some $t \in C(a,b) \setminus (A \cup B)$. $T \setminus t = P \cup Q$, separated sets such that $a \in P$ and $b \in Q$; A is connected, lies in $T \setminus t$ and intersects P, so $A \subset P$. Similarly, $B \subset Q$.

(ii) Let $a \in A \setminus B$ and $b \in B \setminus A$. By 2.11 (i), since $A \cup B$ is connected, $C(a,b) \subset A \cup B$. By 2.10, $C(a,b)$ is connected, so there is some $x \in C(a,b) \cap (A \cap B)$. Therefore $A \cap B$ separates a and b, and hence $A \setminus B$ and $B \setminus A$ since they are connected.

(2.14) Lemma. Let T be a tree and Y a space. Let $R \subset T \times Y$ be such that $T = RY$, R is monotone, noninclusive, and Ry is closed for each y in Y. Let B be the set of all branch points of T and for each b in B, let bR be a single point. If y is a point of Y such that $T \setminus Ry = P \cup Q$, nonnull separated sets, then $(P^*)R \cap QR = \square$.

Proof: Suppose to the contrary that there is some x in $(P^*)R \cap QR$. Then $Rx \cap P^* \neq \square$ and $Rx \cap Q \neq \square$. The latter fact assures us that $x \neq y$. If $Rx \cap Ry = \square$, then $Rx \subset P \cup Q$; $Rx \cap P^* \neq \square$ then implies that $Rx \cap P \neq \square$; and $Rx \cap Q \neq \square$, which imply that Rx is not connected. But R is monotone by hypothesis; hence $Rx \cap Ry$ must

be nonnull. No point of B can lie in $Rx \cap Ry$, since for b in B, bR is a single point and we know $x \neq y$. A tree is hereditarily unicoherent (see Theorem 9 of [21]) and Rx and Ry are subcontinua of T, so $Rx \cap Ry$ is connected. Therefore there is some component J of $T \setminus B$ such that $Rx \cap Ry \subset J$.

Suppose $x \neq y$ in $(J^*)R$ and $Rx \cap J^* \subset Ry \cap J^*$. Then $Rx \not\subset J^*$ since R is noninclusive, and Rx is connected, so there is some $b \in Rx \cap (J^* \setminus J)$; this implies that $b \in B$. But $b \in Rx \cap J^*$ and $Rx \cap J^* \subset Ry \cap J^*$ imply that $b \in Rx \cap Ry$, which is false. Therefore $Rx \cap J^* \not\subset Ry \cap J^*$, and as a corollary, J^* is not degenerate. Then by 2.12, J^* is an arc, so we may suppose a simple order has been defined on J^*. Since a tree is hereditarily unicoherent, we may suppose that $Rx \cap J^* = [p,q]$ and $Ry \cap J^* = [r,s]$. Since neither of these sets contains the other, we may suppose $p < r$ and $q < s$; and because $Rx \cap Ry$ is nonnull and contained in J, we have $p < r \leq q < s$.

By 2.13 (ii), $T \setminus [r,q] = P' \cup Q'$, where P' and Q' are separated sets such that $[p,r) \subset P'$ and $(q,s] \subset Q'$; further, since $[r,q] \subset J \subset T \setminus B$, P' and Q' are connected sets. Now $Rx \cap Q' = \square$ and $Ry \cap P' = \square$; for suppose $Rx \cap Q' \neq \square$. Since $s \in J^*$, $J \cup s$ is connected, so by 2.11 (i), $C(q,s) = [q,s] \subset J \cup s$; hence $(q,s) \subset J$, and since $q \neq s$, $(q,s) \neq \square$. Therefore (q,s) separates T into the sets, $P' \cup [r,q]$ and $Q' \setminus (q,s)$; we know $Rx \subset T \setminus (q,s)$, so (q,s) separates $Rx \cap P'$ from $Rx \cap Q'$. Rx is connected, however, and we know that $p \in Rx \cap P'$, so $Rx \cap Q'$ must be null. Similarly, $Ry \cap P' = \square$.

Therefore, $T \setminus Ry = P' \cup (Q' \setminus Ry)$ and of course these are separated sets. By hypothesis, $T \setminus Ry = P \cup Q$, nonnull separated sets,

and by supposition, $Rx \cap P^* \neq \square$ and $Rx \cap Q \neq \square$. Since $Rx \cap P' \neq \square$ and this is a component of $T \setminus Ry$, it follows that $Rx \cap (Q' \setminus Ry)^* \neq \square$. But $(Q' \setminus Ry)^* \subset Q'$, and $Rx \cap Q' = \square$. We have reached a contradiction, so it follows that no such x can exist.

(2.15) <u>Theorem</u>. Let T be a tree and Y a nondegenerate Hausdorff space. Let $R \subset T \times Y$ be such that $T = RY$, R is point compact, u.s.c. on T, monotone, noninclusive, and $R^{(-1)}$ is monotone. Also, for each branch point b of T, let bR be a single point. Then $TR = Y$ and Y is a tree.

Proof: Since R is noninclusive, by 2.1 (i), $TR = Y$. By 2.5, TR is connected; that is, Y is connected. By 1.17, Y is compact, and by hypothesis, Y is Hausdorff, so Y is a continuum. To complete the proof that Y is a tree, let $x \neq y$ in Y; we need to find some z in Y such that z separates x and y.

Case 1. $Rx \cap Ry = \square$. By 2.2 (ii) and since R is monotone, Rx and Ry are subcontinua of T; then by 2.13 (i), there is some t in T which separates Rx and Ry. Since $T = RY$, tR is not empty, so we can choose z in tR. Neither x nor y lies in tR, so $z \notin (x \cup y)$. Now $t \in Rz$, and since t separates Rx and Ry, Rz separates $Rx \setminus Rz$ and $Ry \setminus Rz$. Since R is noninclusive, neither of these sets is empty. So $T \setminus Rz = P \cup Q$, nonnull separated sets, and $x \in PR$ and $y \in QR$. Since $PR \cup QR = (T \setminus Rz)R$ and since R is noninclusive, by 2.1 (ii), $PR \cup QR = Y \setminus z$. Finally, PR and QR are separated; for by 2.2 (i), $(PR)^* \subset (P^*)R$, and by 2.14, $(P^*)R \cap QR = \square$. Similarly, $PR \cap (Q^*)R = \square$.

Case 2. $Rx \cap Ry \neq \square$. Since T is hereditarily unicoherent (see Theorem 9 of [21]), since R is monotone and Rx and Ry are closed by 2.2 (ii), $Rx \cap Ry$ is a continuum. Since bR is a single point for each branch point b of T, $Rx \cap Ry \subset J$, a component of $T \setminus B$, where B is the set of branch points of T. As in the proof of 2.14, $Rx \cap J^* \not\subset Ry \cap J^*$, hence J^* is not degenerate. Therefore, by 2.12, J^* is an arc. Let the noncutpoints of J^* be a and b, and define $t \leq t'$ in J^* iff $t = a$, $t = t'$ or t separates a and t'.

Let $S = R \cap (J^* \times Y)$, and consider S as a relation in $J^* \times (J^*)R$. It is easily seen that S satisfies all the hypotheses of 2.6, and hence that $(J^*)S = (J^*)R$ is an arc whose noncutpoints are aR and bR. Define $w \leq w'$ in $(J^*)S$ iff $w = aR$, $w = w'$ or w separates aR and w'. Now $(x \cup y) \subset (J^*)S$ and $x \neq y$, so we may suppose $x < y$ and we can find some z such that $x < z < y$. We will now prove that $T \setminus Rz = P \cup Q$, separated sets such that $x \in PR$ and $y \in QR$.

Let $Sx = [p_x, q_x]$, $Sy = [p_y, q_y]$, and $Sz = [p_z, q_z]$. Since $(J^*)S$ is an arc, 2.4 applies to tell us that $p_x < p_z < p_y$ and $q_x < q_z < q_y$. By hypothesis, $Rx \cap Ry = Sx \cap Sy \neq \square$, and S is noninclusive, so we may conclude that $p_x < p_z < p_y \leq q_x < q_z < q_y$. Then $[p_x, q_z]$ and $[p_z, q_y]$ are subcontinua of T which satisfy 2.13 (ii); their intersection is $[p_z, q_z]$, hence $T \setminus [p_z, q_z] = P \cup Q$, separated sets such that $[p_x, p_z) \subset P$ and $(q_z, q_y] \subset Q$. Therefore $x \in PR$ and $y \in QR$. Finally, $PR \cup QR = Y \setminus z$ and PR and QR are separated, just as in Case 1, since Rz connected implies $Rz = [p_z, q_z]$.

Whether or not 2.15 implies that R is l.s.c. on T is unknown. While it is true that any union of relations, each in $T \times Y$ and l.s.c. on T, is l.s.c. on T, even if $R = \bigcup\{R \cap (J^* \times Y) \mid J \text{ is a component of } T \setminus B\}$ (which need not be true), we know only that $R \cap (J^* \times Y)$ is l.s.c. on J^*. As mentioned on p. 12, this does not imply that $R \cap (J^* \times Y)$ is l.s.c. on T.

(2.15') Theorem. Let T be a tree and Y a nondegenerate Hausdorff space. Let $R \subset T \times Y$ be such that $T = RY$, R is point compact, u.s.c. on T, monotone, noninclusive, and R is l.s.c. on T. Also, for each branch point b of T, let bR be a single point. Then $TR = Y$, $R^{(-1)}$ is monotone and Y is a tree.

Proof: By 2.1 (i), $TR = Y$. To prove that $R^{(-1)}$ is monotone, let $t \varepsilon T$. If $t \varepsilon B^*$, let $[b_a \mid a \varepsilon D]$ be a sequence contained in B such that $[b_a]$ converges to t; then $(b_a)R$ is a point for each a in D, and because R is l.s.c. on T, tR is a point also: for suppose $tR \supset (y \cup y')$, where $y \neq y'$, and let U and U' be disjoint neighborhoods of y and y', respectively. Then $t \varepsilon RU \cap R(U')$, and this set is open because R is l.s.c. on T; hence there is some a_0 such that for every a beyond a_0, $b_a \varepsilon RU \cap R(U')$. That is, $(b_a)R \cap U \neq \square$ and $(b_a)R \cap U' \neq \square$. But such b_a exists and $(b_a)R$ is a point, which cannot lie in both the disjoint sets U and U'. Thus tR contains at most one point and is therefore connected. If $t \notin B^*$, then let J be the component of $T \setminus B^*$ such that $t \varepsilon J$; T is locally connected (see [20]) so J is open in T, and hence J is not degenerate. Therefore J^* is an arc,

by 2.12. Let $S = R \cap (J^* \times Y)$ and consider S as a relation in $J^* \times (J^*)R$. It is easy to prove that S satisfies all the hypotheses of 2.6', and hence $S^{(-1)}$ is monotone. That is, for each $t \in J^*$, $tS = tR$ is connected. This completes the proof that $R^{(-1)}$ is monotone.

Now 2.15 can be applied to conclude that Y is a tree.

Let us define a <u>dendrite</u> to be a metric, locally connected continuum in which each two points can be separated by a third point. Ward proved in [20] that a tree is locally connected, and hence that a metric tree is a dendrite.

If $(Y, \leq)$ is a partially ordered set and $A \subset Y$, we will say a is a <u>zero</u> of A iff $a \in A$ and $A \subset M(a)$.

(2.16) <u>Theorem</u>. Let D be a dendrite and Y a nondegenerate Hausdorff space. Let $R \subset D \times Y$ be such that $D = RY$, R is monotone, noninclusive, point compact, and u.s.c. and l.s.c. on D. Also, for each branch point b of D, let bR be a point. Then X is a dendrite.

Proof: By 2.15', Y is a tree and $R^{(-1)}$ is monotone. Define $f\colon D \longrightarrow \overline{2}^{Y}$ by $f(d) = dR$ for each d in D; since R is u.s.c. and l.s.c. on D, f is continuous by 1.8. Since R is point compact and $R^{(-1)}$ is monotone, dR is a continuum for each d in D.

Choose e in Y and define $\leq$ on Y as follows: let $x \leq y$ iff $x = e$, $x = y$ or x separates e and y; this is a semi-continuous order by the theorem of Ward's which we quoted as 2.9. Let $\mathcal{A} = \{A \in \overline{2}^{Y} \mid A$ has a zero$\}$; Capel and Strother proved in [2] that each nonnull

subcontinuum of Y has a zero, hence $f(D) \subset \mathcal{A}$; they also proved there that, since Y is compact, the function $g: \mathcal{A} \longrightarrow Y$ defined by $g(A)$ = zero of A, is continuous.

Therefore, $gf: D \longrightarrow Y$ is continuous. D is a compact metric space, hence $gf(D)$ is also metric (see 3-23 of [7], for example). To complete the proof, we need to show that $gf(D) = Y$. We know $gf(D)$ is connected and Y is irreducibly connected about its set of noncutpoints, so it will suffice to prove that any noncutpoint of Y lies in $gf(D)$.

Let y be any noncutpoint of Y. We will prove that there is some point d in D such that $dR = y$; then $f(d) = y$, and therefore $gf(d) = y$. It was shown in the proof of 2.15' that for each b in B^*, bR is a point; so if $y \in (B^*)R$, we are done. Otherwise, there is some component J of $T \setminus B^*$ such that $y \in (J^*)R$. It was shown in 2.15' that J^* is not degenerate, so by 2.12, J^* is an arc; let a and b be the noncutpoints of J^*. Since y does not cut Y, it cannot cut $(J^*)R$; as was shown in the proof of 2.15, the noncutpoints of $(J^*)R$ are aR and bR, points of Y. Therefore either $y = aR$ or $y = bR$, and we are done.

As was mentioned on p. 31, a relation which preserves unicoherence has not yet been found. We do know that something more than the conditions used so far is necessary. In [14] Strother gives an example of a very well-behaved relation which takes a 2-cell onto its boundary; we quote this example below.

(2.17) Example (Strother). Let X be the unit disc and S the unit circle; that is, S is the boundary of X. Define $R \subset X \times S$ as follows. If x is the origin, let $xR = S$. If x is not the origin: (a) Extend the segment from the origin through x until it meets S in a point A. (b) Draw a perpendicular at x to the radius constructed in (a) and denote its intersections with S as B and C. (c) Consider the closed arc BAC on S. Let $MBACN$ be the closed arc of S with center A, length twice the length of the arc BAC, and having end points M and N. (d) Let $xR = MBACN$.

It is geometrically obvious that R is u.s.c., l.s.c., point compact, monotone, noninclusive, and the inverse relation is monotone; but X is unicoherent and $XR = S$ is not.

REFERENCES

1. R. F. Arens, A Topology for Spaces of Transformations, Annals of Mathematics, vol. 47 (1946), pp. 480-489.
2. C. E. Capel and W. L. Strother, Multi-valued Functions and Partial Order, Portugaliae Mathematica, vol. 17 (1958), pp. 41-47.
3. G. Choquet, Convergences, Annales de l'Universite de Grenoble, vol. 23 (1947-48), pp. 55-112.
4. S. P. Franklin, Concerning Continuous Relations, Dissertation, University of California, Los Angeles, 1963.
5. O. Frink, Topology in Lattices, Transactions of the American Mathematical Society, vol. 51 (1942), pp. 569-582.
6. F. Hausdorff, Set Theory, New York, 1957.
7. J. G. Hocking and G. S. Young, Topology, Reading, Massachusetts, 1961.
8. J. L. Kelley, General Topology, Princeton, 1955.
9. J. W. Keesee, Notes on Euclidean N-space, Department of Mathematics, Tulane University, New Orleans, 1949.
10. C. Kuratowski, Les Fonctions Semi-continues dans l'Espace des Ensembles Fermes, Fundamenta Mathematica, vol. 18 (1932), pp. 148-160.
11. E. Michael, Topologies on Spaces of Subsets, Transactions of the American Mathematical Society, vol. 71 (1951), pp. 152-182.
12. R. L. Moore, Foundations of Point Set Theory, American Mathematical Society Colloquium Publications, vol. 13 (revised), 1962.

13. W. L. Strother, Continuity for Multi-valued Functions and Some Applications to Topology, Dissertation, Tulane University, New Orleans, 1951.

14. ________, On an Open Question Concerning Fixed Points, Proceedings of the American Mathematical Society, vol. 4 (1953), pp. 988-993.

15. ________, Continuous Multi-valued Functions, Boletim da Sociedade de Matematica de Sao Paulo, vol. 10 (1955), pp. 87-120.

16. ________, Fixed Points, Fixed Sets and M-retracts, Duke Mathematical Journal, vol. 22 (1955), pp. 551-556.

17. L. Vietoris, Bereiche Zweiter Ordnung, Monatshefte fur Mathematik und Physik, vol. 33 (1923), pp. 49-62.

18. A. D. Wallace, A Fixed Point Theorem, Bulletin of the American Mathematical Society, vol. 51 (1945), pp. 413-416.

19. L. E. Ward, Jr., Partially Ordered Topological Spaces, Proceedings of the American Mathematical Society, vol. 5 (1954), pp. 144-161.

20. ________, A Note on Dendrites and Trees, Proceedings of the American Mathematical Society, vol. 5 (1954), pp. 992-994.

21. ________, Mobs, Trees and Fixed Points, Proceedings of the American Mathematical Society, vol. 8 (1957), pp. 798-804.

22. G. T. Whyburn, Analytic Topology, American Mathematical Society Colloquium Publications, vol. 28, 1942.

23. R. L. Wilder, Topology of Manifolds, American Mathematical Society Colloquium Publications, vol. 32, 1949.

24. J. W. T. Youngs, Arc-spaces, Duke Mathematical Journal, vol. 7 (1940), pp. 68-84.

BIOGRAPHICAL SKETCH

Jane Maxwell Day was born March 12, 1937, at Avon Park, Florida. She graduated from Avon Park High School in June, 1954. She received a Bachelor of Arts degree with High Honors in January, 1958, and a Master of Science degree in January, 1961, both from the University of Florida. She served as a graduate assistant in the school year 1959-1960 and as an instructor in the school year 1960-1961, in the Department of Mathematics. She was granted a National Science Foundation Cooperative Fellowship for study at the University of Florida from September, 1963, through August, 1964.

Her husband is Walter Ransom Day, Jr., and she has a daughter, Bonnie Claire. She is a member of Phi Beta Kappa and Phi Kappa Phi.

This dissertation was prepared under the direction of the candidate's supervisory committee and has been approved by all members of that committee. It was submitted to the Dean of the College of Arts and Sciences and to the Graduate Council, and was approved as partial fulfillment of the requirements for the degree of Doctor of Philosophy.

April 18, 1964

Dean, College of Arts and Sciences

Dean, Graduate School

Supervisory Committee:

Chairman

C. B. Smith

T. C. Moore